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On $h\alpha$-T_0 and $h\alpha$-T_1 Spaces in Generalized Topological Space

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***Abstract*:** In this paper we introduce continuity, closed mapping, homeomorphism in $h\alpha$-generalized topological space. We also define $D_{\mu_{h\alpha}}$-set in μ-$h\alpha$-generalized topological space and study the relation between $D_{\mu_{h\alpha}}$-set and μ-$h\alpha$-open set. Also we introduce μ-$h\alpha$-T_0 space, μ-$h\alpha$-D_0 space, μ-$h\alpha$-T_1 space and μ-$h\alpha$-D_1space. Their interrelationship, properties and characterizations are obtained.

***Keywords*:** Generalized Topological space, μ-α-Generalized Topological space, μ-$h\alpha$-Generalized Topological space,($\mu_{h\alpha}, \nu_{h\alpha}$)-continuous,($\mu_{h\alpha}, \nu_{h\alpha}$)-homeomorphism, μ-$h\alpha$-T_0 space, μ-$h\alpha$-D_0 space, μ-$h\alpha$-T_1 space and μ-$h\alpha$-D_1 space.

1. Introduction:

N. Levein was the first to broaden topology by replacing open sets with semi-open sets in his article "Semi-open sets and semi-continuity in topological spaces". The notions of α-open set, h-open set, and $h\alpha$-open set were initially introduced by O.Najastad [19], A. Fadhil [10], B.S. Abdullah, Sabh W. Askandar, and Ruquyah N. Balo [1], respectively.

In 2002, A. Csaszar first proposed the concept of generalized topological space. A generalized topology is a collection of subsets of a set that are closed in any arbitrary union. Let X be a non empty set and $\mathcal{P}(X)$be the power set of X.A subfamily $\mu$of$\mathcal{P}(X)$ is called a generalized topology (GT, for short) on X if μ is closed under arbitrary union. $(X, \mu)$is called a generalized topological space (GTS)[7]. μ-open sets are the members of μ, and μ-closed sets are the complement of these. In GTS (X, μ), here $M_\mu = \cup \{U : U \in \mu\}$. A GTS (X, μ) is called strong if $M_\mu = X$[4].

Dr.S.B.Tadam and Ms.K.R.Sharma introduced the idea of μ-$h\alpha$-generalized topological space[20].A subset A of a generalized topological space X is said μ-$h\alpha$-open set denoted by (μ-$h\alpha$-os) if for each set that is not empty U in X, $U \neq X$ and U is μ-α-open such that $A \subseteq i_\mu(A \cup U)$. The collection of all μ-$h\alpha$-open sets is denoted by $\mu_{h\alpha}$. i.e. $\mu_{h\alpha} = \{A : A$ is μ-$h\alpha$-open set in $X\}$. Here, $M_{\mu_{h\alpha}} = \cup \{U : U \in \mu_{h\alpha}\}$.

This work presents the concepts of continuity, closed mapping, and homeomorphism in the $h\alpha$-generalized topological space. In addition, we define $D_{\mu_{h\alpha}}$-set and μ-$h\alpha$-generalized topological space and studies the relationship between $D_{\mu_{h\alpha}}$-set in μ-$h\alpha$-open set. Also we introduce μ-$h\alpha$-T_0 space, μ-$h\alpha$-D_0 space, μ-$h\alpha$-T_1 space and μ-$h\alpha$-D_1 space.Their interrelationship, properties and characterizations are obtained.

2. Preliminaries:

In this section some definitions, basic concepts in generalized topological space have been given.

Definition 2.1:μ-T_0:[4] A generalized topological space (X, μ) is said to be μ-T_0 if for any pair of distinct points $x, y \in M_\mu$, $\exists\mu$-open set containing precisely one of x and y.

Definition 2.2: μ-T_1:[4] A generalized topological space $(X, \mu)$is said to be μ-T_1 if $x, y \in M_\mu$, $x \neq y$ implies the existence of μ-open sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$and $x \notin U_2$ and $y \notin U_1$.

Definition 2.3: $\boldsymbol{D_\mu}$-set:[4, 16] A subset A of $X$is called a $D_{\boldsymbol{\mu}}$-set if there are two μ-open sets U and Vsuch that $U \neq X$ and $A = U - V$.

Remark 2.4:[4] Every μ-open set $A \neq X$ is $D_{\boldsymbol{\mu}}$-open set.

Remark 2.5:A D_μ-set is always contained in M_μ.

Definition 2.6:$\boldsymbol{\mu}$-$\boldsymbol{D_0}$:[4] A generalized topological space(X, μ) is called μ-D_0 if for any pair of distinct points x and y of M_μ $\exists$ a D_μ-set of X containing x but not y or a D_μ-set of Xcontaining y but not x.

Definition 2.7:$\boldsymbol{\mu}$-$\boldsymbol{D_1}$:[4] A generalized topological space(X, μ) is called μ-D_1 if for any pair of distinct points x and y of M_μ $\exists$ twoD_μ-sets $U$and Vsuch that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Definition 2.8: Subspace of GTS:[5] Let (X, μ)be a generalized topological space and X^* be any nonempty subset of X.Then the relative generalized topology for $X^*$is the collection μ^* defined as $\mu^* = \{A^* : A^* = A \cap X^*, A \in \mu\}$. Hence$M_\mu{}^* = \cup \{A^* : A^* \in \mu^*\}$.Here (X^*, μ^*) is called the subspace of a generalized topological space $(X, \mu)$and the members of μ^* are said to be the μ^*-open sets of (X^*, μ^*). The set which is the complement of μ^*-open is called μ^*-closed set of (X^*, μ^*).We denote the collection of all μ^*-closed set of X^* by $\mathcal{F}^*$i.e. $\mathcal{F}^* = \{F^* : F^*$is μ^*-closed in $X^*\}$.

We introduce the μ-α-subspace of a μ-α-generalized topological space andμ-$h\alpha$- subspace of a μ-$h\alpha$-generalized topological space and obtained their properties in [21] as follows.

Definition 2.9:$\boldsymbol{\mu}$-$\boldsymbol{\alpha}$-Subspace of a $\boldsymbol{\mu}$-$\boldsymbol{\alpha}$-generalized topological space X:[21] Let (X, μ)be a generalized topological space and X^* be any non empty subset of$X$. Then $\boldsymbol{\mu}$-$\boldsymbol{\alpha}$-relative generalized topology for $X^*$is the collection$\mu_\alpha{}^*$defined as $\mu_\alpha{}^* = \{A^* : A^* = A \cap X^*, A \in \mu_\alpha\}$. Hence$M_{\mu_\alpha}{}^* = \cup \{A^* : A^* \in \mu_\alpha{}^*\}$.Here $(X^*, \mu_\alpha{}^*)$ is called the μ-α-subspace of a μ-α-generalized topological space (X, μ_α) and the members of $\mu_\alpha{}^*$ are said to be the $\mu_\alpha{}^*$-open sets of $(X^*, \mu_\alpha{}^*)$. The set which is the complement of $\mu_\alpha{}^*$-open is called $\mu_\alpha{}^*$-closed set of $(X^*, \mu_\alpha{}^*)$.We denote the collection of all $\mu_\alpha{}^*$-closed set of X^* by $\mathcal{F}_\alpha{}^*$i.e. $\mathcal{F}_\alpha{}^* = \{F^* : F^*$is $\mu_\alpha{}^*$-closed in $X^*\}$.

The μ-α-subspace $(X^*, \mu_\alpha{}^*)$ of a μ-α-generalized topological space $(X, \mu_\alpha)$is also a generalized topological space. Also, $M_{\mu_\alpha}{}^* \subseteq M_{\mu_\alpha}$.

Definition 2.10:$\boldsymbol{\mu}$-$\boldsymbol{h\alpha}$-Subspace of a $\boldsymbol{\mu}$-$\boldsymbol{h\alpha}$-generalized topological space X:[21] Let (X, μ) be a generalized topological space and X^* be any non empty subset of$X$. Then $\boldsymbol{\mu}$-$\boldsymbol{h\alpha}$-relative generalized topology for $X^*$is the collection $\mu_{h\alpha}{}^*$ defined as $\mu_{h\alpha}{}^* = \{A^* : A^* = A \cap X^*, A \in \mu_{h\alpha}\}$. Hence $M_{\mu_{h\alpha}}{}^* = \cup \{A^* : A^* \in \mu_{h\alpha}{}^*\}$. Here $(X^*, \mu_{h\alpha}{}^*)$ is called the μ-$h\alpha$-subspace of a μ-$h\alpha$-generalized topological space $(X, \mu_{h\alpha})$ and the members of $\mu_{h\alpha}{}^*$ are said to be the $\mu_{h\alpha}{}^*$-open sets of $(X^*, \mu_{h\alpha}{}^*)$. The set which is the complement of $\mu_{h\alpha}{}^*$-open is called $\mu_{h\alpha}{}^*$-closed set of $(X^*, \mu_{h\alpha}{}^*)$.We denote the collection of all $\mu_{h\alpha}{}^*$-closed set of X^* by $\mathcal{F}_{h\alpha}{}^*$i.e. $\mathcal{F}_{h\alpha}{}^* = \{F^* : F^*$is $\mu_{h\alpha}{}^*$-closed in $X^*\}$.

The μ-$h\alpha$-subspace $(X^*, \mu_{h\alpha}{}^*)$ of a μ-$h\alpha$-generalized topological space $(X, \mu_{h\alpha})$is also a generalized topological space. Also, $M_{\mu_{h\alpha}}{}^* \subseteq M_{\mu_{h\alpha}}$.

3. Continuity and Homeomorphism in $\boldsymbol{h\alpha}$-generalized topological space:

Definition 3.1: Let(X, μ) and (Y, ν)be the generalized topological spaces, and let $(X, \mu_{h\alpha})$and $(Y, \nu_{h\alpha})$be its derived μ-$h\alpha$-GTS and ν-$h\alpha$-GTS respectively. A function $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is said to be$(\mu_{h\alpha}, \nu_{h\alpha})$-continuous at a point $x \in X$ if and only if for every ν-$h\alpha$-open set G^* containing $f(x)$ there is μ-$h\alpha$-open set G containing x such that $f(G) \subseteq G^*$.

Further we say that$f$ is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous on a set $E \subseteq X$ if and only if it is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous at each point of E.

Theorem 3.2: A function $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is a $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous if and only if the inverse image of every ν-$h\alpha$-open set in $Y$is μ-$h\alpha$-open set in X.

Proof: Suppose that$f$ is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous on X and G^* is ν-$h\alpha$-open set in Y. Then we have to show that $f^{-1}(G^*)$ is μ-$h\alpha$-open set in X.

i.e. We have to show that $f^{-1}(G^*)$is a neighborhood of each of its points.

Let $x \in f^{-1}(G^*)$.Then $f(x) \in G^*$. As f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous on X. ∴for ν-$h\alpha$-open set G^*in Ycontaining $f(x)$ there is μ-$h\alpha$-open set G in X containing x such that $f(G) \subseteq G^*$.

Now, $f(G) \subseteq G^* \Rightarrow G \subseteq f^{-1}(G^*)$

Also, $f(x) \in G^* \Rightarrow x \in f^{-1}(G^*)$

i.e. for the point $x \in f^{-1}(G^*) \exists$ μ-$h\alpha$-open set G such that $x \in G \subseteq f^{-1}(G^*)$.

Thus $f^{-1}(G^*)$is a neighborhood of a point x.

Thus $f^{-1}(G^*)$is a neighborhood of each of its point.

Hence $f^{-1}(G^*)$ is μ-$h\alpha$-open set in X.

Conversely:

Suppose inverse image of every ν-$h\alpha$-open set in Y is μ-$h\alpha$-open set in X. We have to show that the mapping $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is a $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous on X. Let $x \in X$ and G^* be any ν-$h\alpha$-open set containing$f(x)$. Then by hypothesis, $f^{-1}(G^*)$ is μ-$h\alpha$-open set in X. Hence $f^{-1}(G^*)$is a neighborhood of each of its point.

Also $f(x) \in G^* \Rightarrow x \in f^{-1}(G^*)$.

∴for $x \in f^{-1}(G^*)$there is μ-$h\alpha$-open set G in X such that $x \in G \subseteq f^{-1}(G^*)$.

As $G \subseteq f^{-1}(G^*) \Rightarrow f(G) \subseteq f(f^{-1}(G^*)) \subseteq G^*$ i.e. $f(G) \subseteq G^*$.

Also $x \in G \Rightarrow f(x) \in f(G) \subseteq G^*$.

∴for $x \in X$ and for ν-$h\alpha$-open set G^* containing $f(x)$there exists μ-$h\alpha$-open set G in X containing x such that $f(G) \subseteq G^*$. Hence, f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

Theorem 3.3: A function $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is a $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous if and only if the inverse image of every ν-$h\alpha$-closed set in $Y$is μ-$h\alpha$-closed set in X.

Proof: Suppose f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous. Then we have to show that the inverse image of every ν-$h\alpha$-closed set in Y is μ-$h\alpha$-closed set in X.

Let B be ν-$h\alpha$-closed set in Y. Then we have to show that $f^{-1}(B)$ is μ-$h\alpha$-closed set in X. i.e. we have to show that $X - f^{-1}(B)$ is μ-$h\alpha$-open set in X. As B be ν-$h\alpha$-closed set in $Y \Rightarrow Y - B$ is ν-$h\alpha$-open set in Y. Since f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous then by theorem 3.2, $f^{-1}(Y - B)$ is μ-$h\alpha$-open set in X.

Now $f^{-1}(Y - B) = \{x \in X : f(x) = y, y \in Y - B\} = \{x \in X : f(x) = y, y \in Y - B \subseteq M_{\nu_{h\alpha}}\}$

Also, $f^{-1}(B) = \{x \in X : f(x) = y, y \in B\}$

$\therefore X - f^{-1}(B) = \{x \in X : x \notin f^{-1}(B)\} = \{x \in X : \nexists\, y \in B \text{such that } f(x) = y\} = \{x \in X : f(x) = y, y \in Y - B \subseteq M_{\nu_{h\alpha}}\}$

Hence, $f^{-1}(Y - B) = X - f^{-1}(B)$.

As $f^{-1}(Y - B)$ is μ-$h\alpha$-open in X and hence, $X - f^{-1}(B)$ is μ-$h\alpha$-open in X. $\Rightarrow f^{-1}(B)$is μ-$h\alpha$-closed in X.

Conversely: Suppose inverse image of every ν-$h\alpha$-closed set in $Y$is μ-$h\alpha$-closed set in X. We have to show that f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

i.e. we have to show that inverse image of every ν-$h\alpha$-open set in $Y$is μ-$h\alpha$-open set in X. Let A be ν-$h\alpha$-open set in $Y \Rightarrow Y - A$ is ν-$h\alpha$-closed in Y.

$\therefore f^{-1}(Y-A)$ is μ-$h\alpha$-closed set in X. But $f^{-1}(Y-A)$=$X-f^{-1}(A)$.

$\therefore\ X-f^{-1}(A)$ is μ-$h\alpha$-closed set in $X \Rightarrow f^{-1}(A)$ is μ-$h\alpha$-open set in X.

$\therefore$inverse image, $f^{-1}(A)$ is μ-$h\alpha$-open in X for ν-$h\alpha$-open set A in Y.

Hence f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

Theorem 3.4: Let $f:(X,\mu_{h\alpha}) \to (Y,\nu_{h\alpha})$ be a mapping from μ-$h\alpha$-GTS $(X,\mu_{h\alpha})$ to a ν-$h\alpha$-GTS $(Y,\nu_{h\alpha})$. Then the following statements are equivalent.

(1) f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.
(2) $f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}(f(A))$for every $A \subseteq X$.
(3) $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$for every $B \subseteq Y$.

Proof:$(\mathbf{1}) \Leftrightarrow (\mathbf{2})$

Suppose f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous. Let A be any non empty subset of X.

Then we have to prove that $f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}(f(A))$.

We know, $A \subseteq f^{-1}(f(A))$.

Also, $f(A) \subseteq c_{\nu_{h\alpha}}(f(A)) \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(c_{\nu_{h\alpha}}\big(f(A)\big))$.

Hence, $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(c_{\nu_{h\alpha}}\big(f(A)\big))$.

But $c_{\nu_{h\alpha}}\big(f(A)\big)$ is ν-$h\alpha$-closed set in Y and f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

$\therefore f^{-1}(c_{\nu_{h\alpha}}\big(f(A)\big))$ is μ-$h\alpha$-closed set in X containing A.

But $c_{\mu_{h\alpha}}(A)$ is the smallest μ-$h\alpha$-closed set in X containing A.

$\therefore c_{\mu_{h\alpha}}(A) \subseteq f^{-1}(c_{\nu_{h\alpha}}\big(f(A)\big)) \Rightarrow f(c_{\mu_{h\alpha}}(A)) \subseteq f(f^{-1}\left(c_{\nu_{h\alpha}}\big(f(A)\big)\right)) \subseteq c_{\nu_{h\alpha}}(f(A))$.

$\Rightarrow f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}(f(A))$.

Conversely: Now we show that f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous by showing that inverse image of every ν-$h\alpha$-closed set in Y is μ-$h\alpha$-closed set in X.

Let B be ν-$h\alpha$-closed set in Y.

$\therefore B = c_{\nu_{h\alpha}}(B)$. Denote, $f^{-1}(B) = A \subseteq X$.

By hypothesis, for the subset A of X we have, $f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}(f(A))$.

i.e. $f\left(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)\right) \subseteq c_{\nu_{h\alpha}}\left(f\big(f^{-1}(B)\big)\right) \subseteq c_{\nu_{h\alpha}}(B) = B$.

$\therefore f\left(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)\right) \subseteq B$.

$\Rightarrow f^{-1}(f\left(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)\right)) \subseteq f^{-1}(B)$.

$\therefore c_{\mu_{h\alpha}}\big(f^{-1}(B)\big) \subseteq f^{-1}(B)$.

Also, $f^{-1}(B) \subseteq c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)$.

Hence, $f^{-1}(B) = c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)$, μ-$h\alpha$-closed set in X.

Thus inverse image of ν-$h\alpha$-closed set in Y is μ-$h\alpha$-closed set in X. Hence f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

$(\mathbf{2}) \Leftrightarrow (\mathbf{3})$

Suppose that, $f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}(f(A))$ for every $A \subseteq X$.

To prove that $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$ for every $B \subseteq Y$

Let B be any nonempty subset of Y and denote, $f^{-1}(B) = A \subseteq X$.

By hypothesis, $f\left(c_{\mu_{h\alpha}}(A)\right) \subseteq c_{\nu_{h\alpha}}\big(f(A)\big) = c_{\nu_{h\alpha}}\left(f\big(f^{-1}(B)\big)\right) \subseteq c_{\nu_{h\alpha}}(B)$

$\therefore f\left(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)\right) \subseteq c_{\nu_{h\alpha}}(B)$

$\therefore f^{-1}(f\left(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)\right)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$

$\Rightarrow c_{\mu_{h\alpha}}\big(f^{-1}(B)\big) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$.

Conversely: Suppose for any subset $B$of Y, $c_{\mu_{h\alpha}}\big(f^{-1}(B)\big) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$. Let A be any nonempty subset of X and denote $f(A) = B$.

By hypothesis, $c_{\mu_{h\alpha}}\big(f^{-1}(B)\big) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$

i.e. $c_{\mu_{h\alpha}}\big(f^{-1}(f(A))\big) \subseteq f^{-1}(c_{\nu_{h\alpha}}(f(A)))$

$\Rightarrow c_{\mu_{h\alpha}}(A) \subseteq c_{\mu_{h\alpha}}\big(f^{-1}(f(A))\big) \subseteq f^{-1}(c_{\nu_{h\alpha}}(f(A)))$

$\Rightarrow c_{\mu_{h\alpha}}(A) \subseteq f^{-1}(c_{\nu_{h\alpha}}(f(A)))$

$\Rightarrow f(c_{\mu_{h\alpha}}(A)) \subseteq f(f^{-1}(c_{\nu_{h\alpha}}\big(f(A)\big))) \subseteq c_{\nu_{h\alpha}}\big(f(A)\big)$

i.e. $f(c_{\mu_{h\alpha}}(A)) \subseteq c_{\nu_{h\alpha}}\big(f(A)\big)$.

$(1) \Leftrightarrow (3)$

Suppose for any subset B of Y, $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$. Now we show f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous by showing that inverse image of ν-$h\alpha$-closed set in $Y$is μ-$h\alpha$-closed set in X.

Let $B \subseteq Y$ be ν-$h\alpha$-closed set. By hypothesis, $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$.

To show that $f^{-1}(B)$ is μ-$h\alpha$-closed set in X we have to show that, $c_{\mu_{h\alpha}}(f^{-1}(B)) = f^{-1}(B)$.

We know, $f^{-1}(B) \subseteq c_{\mu_{h\alpha}}(f^{-1}(B))$

By hypothesis, $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}\left(c_{\nu_{h\alpha}}(B)\right) = f^{-1}(B)$

Hence, $c_{\mu_{h\alpha}}(f^{-1}(B)) = f^{-1}(B)$

$\Rightarrow f^{-1}(B)$ is μ-$h\alpha$-closed set in X.

Hence, inverse image of ν-$h\alpha$-closed set in Y is μ-$h\alpha$-closed set in X.

Hence, f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous.

Conversely: Suppose f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous and B is any nonempty subset of Y. Then we have to show that $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$. Denote $f^{-1}(B) = A$.

As f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous thus from above $f(c_{\mu_{h\alpha}}(A)) \subseteq (c_{\nu_{h\alpha}}(f(A))$.

i.e. $f(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)) \subseteq c_{\nu_{h\alpha}}(f(f^{-1}(B)))$.

Hence,$f(c_{\mu_{h\alpha}}\big(f^{-1}(B)\big)) \subseteq c_{\nu_{h\alpha}}\left(f\big(f^{-1}(B)\big)\right) \subseteq c_{\nu_{h\alpha}}(B)$.

Thus, $f^{-1}(f(c_{\mu_{h\alpha}}(f^{-1}(B))) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$.

i.e.$c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(f(c_{\mu_{h\alpha}}(f^{-1}(B))) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$.

Hence, $c_{\mu_{h\alpha}}(f^{-1}(B)) \subseteq f^{-1}(c_{\nu_{h\alpha}}(B))$.

Theorem 3.5: If $f: (X, \mu_{h\alpha}) \rightarrow (Y, \nu_{h\alpha})$ is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous then $f(X - M_{\mu_{h\alpha}}) \subseteq Y - M_{\nu_{h\alpha}}$.

Proof: Let $y \in f(X - M_{\mu_{h\alpha}})$. Then $y = f(x)$ for some $x \in X - M_{\mu_{h\alpha}}$.

Now, $x \in X - M_{\mu_{h\alpha}} \Rightarrow x \in X$ but $x \notin M_{\mu_{h\alpha}} = \cup \{A : A \in \mu_{h\alpha}\}$

i.e. $x \notin A$ for all $A \in \mu_{h\alpha}$.

As $M_{\nu_{h\alpha}}$ is ν-$h\alpha$-open set and the mapping $f: (X, \mu_{h\alpha}) \rightarrow (Y, \nu_{h\alpha})$ is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous thus $f^{-1}(M_{\nu_{h\alpha}})$ is μ-$h\alpha$-open set. i.e. $f^{-1}(M_{\nu_{h\alpha}}) \in \mu_{h\alpha}$.

Hence, $x \notin f^{-1}(M_{\nu_{h\alpha}})$

$\Rightarrow f(x) = y \notin M_{\nu_{h\alpha}}$

$\Rightarrow y \in Y - M_{\nu_{h\alpha}}$

i.e. for any $y \in f(X - M_{\mu_{h\alpha}}) \Rightarrow y \in Y - M_{\nu_{h\alpha}} \Rightarrow f(X - M_{\mu_{h\alpha}}) \subseteq Y - M_{\nu_{h\alpha}}$.

Remark 3.6: From above theorem we have $f(X - M_{\mu_{h\alpha}}) \subseteq Y - M_{\nu_{h\alpha}}$.

Taking the inverse on both sides we get, $f^{-1}(f(X - M_{\mu_{h\alpha}})) \subseteq f^{-1}(Y - M_{\nu_{h\alpha}})$

i.e.$X - M_{\mu_{h\alpha}} \subseteq f^{-1}(f(X - M_{\mu_{h\alpha}})) \subseteq f^{-1}(Y - M_{\nu_{h\alpha}})$

$\Rightarrow X - M_{\mu_{h\alpha}} \subseteq f^{-1}(Y - M_{\nu_{h\alpha}})$.

Definition 3.7: A mapping $f: (X, \mu_{h\alpha}) \rightarrow (Y, \nu_{h\alpha})$ is said to be $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism if f is bijective, f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous and f^{-1} is $(\nu_{h\alpha}, \mu_{h\alpha})$-continuous.

A property of sets which is preserved by $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism is called a $(\mu_{h\alpha}, \nu_{h\alpha})$-topological property.

Definition 3.8: A mapping $f: (X, \mu_{h\alpha}) \rightarrow (Y, \nu_{h\alpha})$ is said to be $(\mu_{h\alpha}, \nu_{h\alpha})$-closed if the image of every μ-$h\alpha$-closed set in X is ν-$h\alpha$-closed set in Y.

Remark 3.9: It follows from the above definition that f is $(\mu_{h\alpha}, \nu_{h\alpha})$-closed if and only if $c_{\nu_{h\alpha}}(f(A)) \subseteq f(c_{\mu_{h\alpha}}(A))$.

4. $h\alpha$-T_0 Space and $h\alpha$-D_0 Space:

Definition 4.1: μ-$h\alpha$-T_0 Space: A generalized topological space (X, μ) is said to be μ-$h\alpha$-T_0if and only if for any $x, y \in M_{\mu_{h\alpha}}$, $x \neq y$, $\exists$ μ-$h\alpha$-open set A such that $x \in A, y \notin A$ or $y \in A, x \notin A$.

We shall refer to μ-$h\alpha$-T_0 space for a particular GTS μ in order to prevent ambiguity in $h\alpha$-T_0 space with regard to GTS.

Theorem 4.2: Let X be a μ-$h\alpha$-T_0 space and X^* be a nonempty subset of X. Then the μ-$h\alpha$-subspace $(X^*, {\mu_{h\alpha}}^*)$ is also a μ-$h\alpha$-T_0 space.(i.e. $h\alpha$-T_0 is a hereditary property.)

Proof: Let $x, y \in {M_{\mu_{h\alpha}}}^*$ such that $x \neq y$.

But ${M_{\mu_{h\alpha}}}^* \subseteq M_{\mu_{h\alpha}} \Rightarrow x, y \in M_{\mu_{h\alpha}}$ with $x \neq y$. As X is μ-$h\alpha$-T_0space. Thus for $x, y \in M_{\mu_{h\alpha}}$with $x \neq y$ there exists $A \in \mu_{h\alpha}$ such that either $x \in A, y \notin A$ or $y \in A, x \notin A$.

Suppose $x \in A$ and $y \notin A$. Now as $x, y \in M_{\mu_{h\alpha}}{}^{*} \subseteq X^{*} \Rightarrow x, y \in X^{*}$.

Thus we get, $x \in A \cap X^{*}$ but $y \notin A \cap X^{*}$.

As $A \in \mu_{h\alpha} \Rightarrow A \cap X^{*} \in \mu_{h\alpha}{}^{*}$.

Thus for any $x, y \in M_{\mu_{h\alpha}}{}^{*}$ with $x \neq y$ there exists $A \cap X^{*} \in \mu_{h\alpha}{}^{*}$ such that $x \in A \cap X^{*}$ but $y \notin A \cap X^{*}$.

$\Rightarrow (X^{*}, \mu_{h\alpha}{}^{*})$ is a μ-$h\alpha$-T_0 space.

Thus every μ-$h\alpha$-subspace of a μ-$h\alpha$-T_0 space is also μ-$h\alpha$-T_0.

Hence μ-$h\alpha$-T_0 is a hereditary property.

Theorem 4.3: If $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is a $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism and $(X, \mu_{h\alpha})$ is a μ-$h\alpha$-T_0 space then $(Y, \nu_{h\alpha})$is a ν-$h\alpha$-T_0 space. (i.e. $h\alpha$-T_0 is a $(\mu_{h\alpha}, \nu_{h\alpha})$-topological property.)

Proof: Let $y_1, y_2 \in M_{\nu_{h\alpha}}$ with $y_1 \neq y_2$.

For $y_1, y_2 \in M_{\nu_{h\alpha}} \subseteq Y \Rightarrow y_1, y_2 \in Y = f(X)$.

$\Rightarrow y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$.

As the mapping f is bijective and$y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.

Also, $y_1, y_2 \in M_{\nu_{h\alpha}} \Rightarrow y_1, y_2 \notin Y - M_{\nu_{h\alpha}}$ i.e. $f(x_1), f(x_2) \notin Y - M_{\nu_{h\alpha}}$.

The mapping f is $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism and hence, $f\left(X - M_{\mu_{h\alpha}}\right) \subseteq Y - M_{\nu_{h\alpha}}$.

$\therefore f(x_1), f(x_2) \notin f\left(X - M_{\mu_{h\alpha}}\right) \Rightarrow x_1, x_2 \notin X - M_{\mu_{h\alpha}}$.

$\Rightarrow x_1, x_2 \in M_{\mu_{h\alpha}}, x_1 \neq x_2$ and X is a μ-$h\alpha$-T_0 space.

Hence there exists μ-$h\alpha$-open set A containing $x_1$but not x_2 or containing x_2 but not x_1.

As A is μ-$h\alpha$-open set and the mapping f is a $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism, hence $f(A)$ is ν-$h\alpha$-open set containing $f(x_1)$ but not $f(x_2)$ or containing $f(x_2)$ but not $f(x_1)$.

Hence, $(Y, \nu_{h\alpha})$is a ν -$h\alpha$-T_0 space. i.e. $h\alpha$-T_0 is a $(\mu_{h\alpha}, \nu_{h\alpha})$-topological property.

Definition 4.4: $D_{\mu_{h\alpha}}$-set: A subset A of X is called a $D_{\mu_{h\alpha}}$-set if there are two μ-$h\alpha$-open sets U and V such that $U \neq X$ and $A = U - V$.

Remark 4.5: Every μ-$h\alpha$-open set is $D_{\mu_{h\alpha}}$-set.

Definition 4.6: μ-$h\alpha$-D_0 Space: A generalized topological space (X, μ) is said to be μ-$h\alpha$-D_0 if and only if for any $x, y \in M_{\mu_{h\alpha}}, x \neq y, \exists\ D_{\mu_{h\alpha}}$-set A containing one of them but not the other.

We shall refer to μ-$h\alpha$-D_0 space for a particular GTS μ in order to prevent ambiguity in $h\alpha$-D_0 space with regard to GTS.

Proposition 4.7: Every D_μ-set is $D_{\mu_{h\alpha}}$-set.

Proof: Let A be a D_μ-set of X. Thus by definition, there are two μ-open sets U and V such that $U \neq X$ and $A = U - V$.

As $U, V \in M_\mu$ and $M_\mu \subseteq M_{\mu_{h\alpha}}$.

$\Rightarrow U, V \in M_{\mu_{h\alpha}}$ such that $U \neq X$ and $A = U - V$.

$\Rightarrow A$ is $D_{\mu_{h\alpha}}$-set in X.

Thus every D_{μ}-set is $D_{\mu_{h\alpha}}$-set.

Remark 4.8: The converse of above proposition is not true. We prove it by giving a counter example.

Example 4.9: Let $X = \{1,3,5\}$ and $\mu = \{\emptyset, \{1,3\}, \{1,5\}, X\}$, $\mu_{\alpha} = \{\emptyset, \{1,3\}, \{1,5\}, X\}$, $\mu_{h\alpha} = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, X\}$

Here $A = \{1,3\} - \{1\} = \{3\}$ is $D_{\mu_{h\alpha}}$-set but not a D_{μ}-set. Here $\{1,3\}, \{1,5\}$ are D_{μ}-sets which are also $D_{\mu_{h\alpha}}$-sets. But there exists $D_{\mu_{h\alpha}}$-sets which are not D_{μ}-sets.

Proposition 4.10: A generalized topological space (X, μ) is μ-$h\alpha$-T_0 if and only if it is μ-$h\alpha$-D_0.

Proof: Let X be a μ-$h\alpha$-T_0 space and $x, y \in M_{\mu_{h\alpha}}$, with $x \neq y$ then by definition of μ-$h\alpha$-T_0, $\exists \mu$-$h\alpha$-open set U such that $x \in U$ but $y \notin U$ or $y \in U$ but $x \notin U$.

Suppose $x \in U$ but $y \notin U$. As U is μ-$h\alpha$-open set $\Rightarrow U$ is $D_{\mu_{h\alpha}}$-set. i.e. $\exists$ $D_{\mu_{h\alpha}}$-set$U$ such that $\in U$, $y \notin U$ or $y \in U$, $x \notin U$.

$\Rightarrow X$ is a μ-$h\alpha$-D_0 space.

Conversely: Let(X, μ) be μ-$h\alpha$-D_0space and $x, y \in M_{\mu_{h\alpha}}$ with $x \neq y$. Then by definition of μ-$h\alpha$-D_0, $\exists$ $D_{\mu_{h\alpha}}$-set A such that $x \in A, y \notin A$ or $y \in A, x \notin A$.

As A is $D_{\mu_{h\alpha}}$-set thus $\exists$ $U, V \in \mu_{h\alpha}$ such that $U \neq X$ and $A = U - V$.

Now $x \in A \Rightarrow x \in U - V \Rightarrow x \in U$ and $x \notin V$ and $y \notin A = U - V = U \cap \complement V$

$\Rightarrow y \in \complement(U \cap \complement V) = \complement U \cup V$

$\Rightarrow y \in \complement U$ or $y \in V$

If $y \in \complement U \Rightarrow y \notin U$ or if $y \in V \Rightarrow y \in V$and $x \notin V \Rightarrow X$ is a μ-$h\alpha$-T_0space.

Theorem 4.11: A generalized topological space (X, μ) is μ-$h\alpha$-T_0 if and only if each pair of distinct points $x, y \in M_{\mu_{h\alpha}}$, $c_{\mu_{h\alpha}}(\{x\}) \neq c_{\mu_{h\alpha}}(\{y\})$.

Proof: Suppose that in X for any $x, y \in M_{\mu_{h\alpha}}$, with $x \neq y$, $c_{\mu_{h\alpha}}(\{x\}) \neq c_{\mu_{h\alpha}}(\{y\})$. So there exists $z \in X$ such that z is contained in one of them but not the other.

Let us suppose that, $z \in c_{\mu_{h\alpha}}(\{x\})$ but $z \notin c_{\mu_{h\alpha}}(\{y\})$. If we had $x \in c_{\mu_{h\alpha}}(\{y\})$ then $c_{\mu_{h\alpha}}(\{x\}) \subseteq c_{\mu_{h\alpha}}\left(c_{\mu_{h\alpha}}(\{y\})\right) = c_{\mu_{h\alpha}}(\{y\})$.[(by theorem 4.16 and by remark 4.12)[20]]

$\Rightarrow c_{\mu_{h\alpha}}(\{x\}) \subseteq c_{\mu_{h\alpha}}(\{y\})$

$\Rightarrow z \in c_{\mu_{h\alpha}}(\{y\})$. This gives contradiction.

Thus, $x \notin c_{\mu_{h\alpha}}(\{y\}) \Rightarrow x \in \complement(c_{\mu_{h\alpha}}(\{y\}))$ i.e $\complement(c_{\mu_{h\alpha}}(\{y\}))$ is μ-$h\alpha$-open set containing x but not y. Thus, X is μ-$h\alpha$-T_0 space.

Conversely:

Let X be a μ-$h\alpha$-T_0 space. Let $x, y \in M_{\mu_{h\alpha}}$ such that $x \neq y$. As X is μ-$h\alpha$-T_0thus there exists μ-$h\alpha$-open set say U containing one of them but not the other.

Suppose $x \in U, y \notin U \Rightarrow y \in \complement U$. As, U is μ-$h\alpha$-open set $\Rightarrow \complement U$ is μ-$h\alpha$-closed set containing y but not x. But $c_{\mu_{h\alpha}}(\{y\})$ is the smallest μ-$h\alpha$-closed set containing y. Hence $c_{\mu_{h\alpha}}(\{y\}) \subseteq \complement U$.

As, $x \notin \complement U \Rightarrow x \notin c_{\mu_{h\alpha}}(\{y\})$

But $x \in c_{\mu_{h\alpha}}(\{x\}) \Rightarrow c_{\mu_{h\alpha}}(\{x\}) \neq c_{\mu_{h\alpha}}(\{y\})$.

Remark 4.12: From proposition 4.10 and from the theorem 4.11, it is observed that "A generalized topological space (X, μ) is μ-$h\alpha$-D_0 if and only if each pair of distinct points $x, y \in M_{\mu_{h\alpha}}$, $c_{\mu_{h\alpha}}(\{x\}) \neq c_{\mu_{h\alpha}}(\{y\})$."

5. $h\alpha$-T_1 space and $h\alpha$-D_1 space:

Definition 5.1: μ-$h\alpha$-T_1 Space: A generalized topological space (X, μ) is said to be μ-$h\alpha$-T_1if and only if for any $x, y \in M_{\mu_{h\alpha}}$, $x \neq y$, $\exists$ two μ-$h\alpha$-open sets U and V such that $x \in U, y \in V$ but $y \notin U$ and $, x \notin V$. i.e. $x \in U - V$ and $y \in V - U$.

We shall refer to μ-$h\alpha$-T_1 space for a particular GTS μ in order to prevent ambiguity in $h\alpha$-T_1 space with regard to GTS.

Theorem 5.2: Let X be a μ-$h\alpha$-T_1 space and X^* be a nonempty subset of X. Then the μ-$h\alpha$-subspace $(X^*, {\mu_{h\alpha}}^*)$ is also a μ-$h\alpha$-T_1 space.(i.e. $h\alpha$-T_1 is a hereditary property.)

Proof: Let $x, y \in {M_{\mu_{h\alpha}}}^*$ such that $x \neq y$.

But ${M_{\mu_{h\alpha}}}^* \subseteq M_{\mu_{h\alpha}} \Rightarrow x, y \in M_{\mu_{h\alpha}}$ with $x \neq y$. As X is μ-$h\alpha$-T_1space. Thus for $x, y \in M_{\mu_{h\alpha}}$with $x \neq y$ there exists $A, B \in \mu_{h\alpha}$ such that $x \in A - B$, and $y \in B - A$.

As $x, y \in {M_{\mu_{h\alpha}}}^* \subseteq X^* \Rightarrow x, y \in X^*$.

$\Rightarrow x \in A \cap X^*$ but $y \notin A \cap X^*$ and $y \in B \cap X^*$ but $x \notin B \cap X^*$.

As $A, B \in \mu_{h\alpha} \Rightarrow A \cap X^*, B \cap X^* \in {\mu_{h\alpha}}^*$.

Thus for any $x, y \in {M_{\mu_{h\alpha}}}^*$ with $x \neq y$ there exists $A, B \in \mu_{h\alpha}$ such that $A \cap X^*, B \cap X^* \in {\mu_{h\alpha}}^*$ and $x \in A \cap X^*$ but $y \notin A \cap X^*$ and $y \in B \cap X^*$ but $x \notin B \cap X^*$.

$\Rightarrow (X^*, {\mu_{h\alpha}}^*)$ is a μ-$h\alpha$-T_1 space.

Thus every μ-$h\alpha$-subspace of a μ-$h\alpha$-T_1 space is also μ-$h\alpha$-T_1.

Hence μ-$h\alpha$-T_1 is a hereditary property.

Theorem 5.3: If $f: (X, \mu_{h\alpha}) \to (Y, \nu_{h\alpha})$ is a $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism and $(X, \mu_{h\alpha})$ is a μ-$h\alpha$-T_1 space then $(Y, \nu_{h\alpha})$is a ν -$h\alpha$-T_1 space. (i.e. $h\alpha$-T_1 is a $(\mu_{h\alpha}, \nu_{h\alpha})$-topological property.)

Proof: Let $y_1, y_2 \in M_{\nu_{h\alpha}}$ with $y_1 \neq y_2$.

For $y_1, y_2 \in M_{\nu_{h\alpha}} \subseteq Y \Rightarrow y_1, y_2 \in Y = f(X)$.

$\Rightarrow y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$.

As the mapping f is bijective and $y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.

Also, $y_1, y_2 \in M_{\nu_{h\alpha}} \Rightarrow y_1, y_2 \notin Y - M_{\nu_{h\alpha}}$ i.e. $f(x_1), f(x_2) \notin Y - M_{\nu_{h\alpha}}$.

The mapping f is $(\mu_{h\alpha}, \nu_{h\alpha})$-continuous and hence $f\left(X - M_{\mu_{h\alpha}}\right) \subseteq Y - M_{\nu_{h\alpha}}$.

$\therefore f(x_1), f(x_2) \notin f\left(X - M_{\mu_{h\alpha}}\right) \Rightarrow x_1, x_2 \notin X - M_{\mu_{h\alpha}}$.

$\Rightarrow x_1, x_2 \in M_{\mu_{h\alpha}}, x_1 \neq x_2$ and X is a μ-$h\alpha$-T_1 space.

Hence, there exists twoμ-$h\alpha$-open sets say$U$ and V such that $x \in U - V$ and $y \in V - U$. As U and V are μ-$h\alpha$-open sets and the mapping f is a $(\mu_{h\alpha}, \nu_{h\alpha})$-homeomorphism, hence $f(U), f(V)$are ν-$h\alpha$-open sets such that $f(x) \in f(U) - f(V)$ and $f(y) \in f(V) - f(U)$.

Hence, $(Y,\nu_{h\alpha})$ is a ν -$h\alpha$-T_1 space. i.e. $h\alpha$-T_1 is a $(\mu_{h\alpha},\nu_{h\alpha})$-topological property.

Definition 5.4: μ-$h\alpha$-D_1 Space: A generalized topological space (X,μ) is said to be μ-$h\alpha$-D_1if and only if for any $x,y\in M_{\mu_{h\alpha}}$, with $x\neq y\exists$ two $D_{\mu_{h\alpha}}$-sets say$U$ and Vsuch that $x\in U, y\in V$ but $y\notin U$ and $x\notin V$. i.e. $x\in U-V$ and $y\in V-U$.

We shall refer to μ-$h\alpha$-D_1 space for a particular GTS μ in order to prevent ambiguity in $h\alpha$-D_1 space with regard to GTS.

Proposition 5.5: If a generalized topological space (X,μ) is μ-$h\alpha$-T_1 then it is μ-$h\alpha$-D_1.

Proof: Let (X,μ) be μ-$h\alpha$-T_1 space and $x,y\in M_{\mu_{h\alpha}}$, with $x\neq y$. As X is μ-$h\alpha$-T_1 then by definition of μ-$h\alpha$-T_1, $\exists$ two μ-$h\alpha$-open sets say$U$and V such that $x\in U-V$ and $y\in V-U$.

But every μ-$h\alpha$-open set is $D_{\mu_{h\alpha}}$-set. Hence$U$and V are $D_{\mu_{h\alpha}}$-sets such that $x\in U-V$ and $y\in V-U$. Hence X is μ-$h\alpha$-D_1.

Remark 5.6: But the converse of above proposition is not true.

Theorem 5.7: A generalized topological space (X,μ) is μ-$h\alpha$-T_1if and only if for each $x\in M_{\mu_{h\alpha}}$, $\{x\}\cup(X-M_{\mu_{h\alpha}})$ is μ-$h\alpha$-closed.

Proof: Suppose, (X,μ) is μ-$h\alpha$-T_1. Let $x\in M_{\mu_{h\alpha}}$. If $M_{\mu_{h\alpha}}=\{x\}$ then $\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)=X$ is μ-$h\alpha$-closed.

Now suppose, $M_{\mu_{h\alpha}}-\{x\}\neq\emptyset$. Thus for every $y\in M_{\mu_{h\alpha}}-\{x\}=M_{\mu_{h\alpha}}\cap\complement\{x\}$ and as X is μ-$h\alpha$-T_1 there exists two μ-$h\alpha$-open sets $U$and V such that $x\in U-V$ and $y\in V-U$.

Now $x\notin V$ i.e. $x\in\complement V\Rightarrow\{x\}\subseteq\complement V\Rightarrow V\subseteq\complement\{x\}$. Hence $y\in V\subseteq\complement\{x\}$.

As V is μ-$h\alpha$-open set $\Rightarrow V\subseteq M_{\mu_{h\alpha}}\Rightarrow V\subseteq M_{\mu_{h\alpha}}\cap\complement\{x\}$.

i.e. for any $y\in M_{\mu_{h\alpha}}\cap\complement\{x\}$ there exists μ-$h\alpha$-open set V such that $y\in V\subseteq M_{\mu_{h\alpha}}\cap\complement\{x\}$.

$\therefore\bigcup_{y\neq x}\{y\}\subseteq\bigcup_{y\neq x}V\subseteq M_{\mu_{h\alpha}}\cap\complement\{x\}\Rightarrow\bigcup_{y\neq x}V=M_{\mu_{h\alpha}}\cap\complement\{x\}$.

But $M_{\mu_{h\alpha}}\cap\complement\{x\}=\complement\left(X-M_{\mu_{h\alpha}}\right)\cap\complement\{x\}=\complement(\left(X-M_{\mu_{h\alpha}}\right)\cup\{x\})$.

Hence $\complement\left(\left(X-M_{\mu_{h\alpha}}\right)\cup\{x\}\right)=\bigcup_{y\neq x}V$.

But $\bigcup_{y\neq x}V$, arbitrary union of μ-$h\alpha$-open sets and hence μ-$h\alpha$-open set.[20]

Thus $\complement\left(\left(X-M_{\mu_{h\alpha}}\right)\cup\{x\}\right)$ is μ-$h\alpha$-open set.

$\Rightarrow\left(\left(X-M_{\mu_{h\alpha}}\right)\cup\{x\}\right)$ is μ-$h\alpha$-closed set.

Conversely:

Suppose for each $x\in M_{\mu_{h\alpha}}$, $\left(\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$ is μ-$h\alpha$-closed set.

We have to show that X is μ-$h\alpha$-T_1.

Let $x,y\in M_{\mu_{h\alpha}}$ such that $x\neq y$. Then $\left(\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$ and $\left(\{y\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$are μ-$h\alpha$-closed sets.

Now $y\in M_{\mu_{h\alpha}}\Rightarrow y\notin X-M_{\mu_{h\alpha}}$. Also $x\neq y\Rightarrow y\notin\{x\}$ i.e. $y\notin\left(\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$.

$\Rightarrow y\in\complement\left(\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$, μ-$h\alpha$-open set.

Also, $x\in\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\Rightarrow x\notin\complement\left(\{x\}\cup\left(X-M_{\mu_{h\alpha}}\right)\right)$.

Thus, we get $C\left(\{x\} \cup \left(X - M_{\mu_{h\alpha}}\right)\right)$ is μ-$h\alpha$-open set containing y but not x.

Similarly, $C\left(\{y\} \cup \left(X - M_{\mu_{h\alpha}}\right)\right)$ is μ-$h\alpha$-open set containing x but not y.

Hence X is μ-$h\alpha$-T_1 space.

6. Relations:

In the above sections, we see the relations between D_μ-set and $D_{\mu_{h\alpha}}$-set, μ-$h\alpha$-T_0 and μ-$h\alpha$-D_0, μ-$h\alpha$-T_1 and μ-$h\alpha$-D_1. Obviously from the definitions 4.6, 5.4, 4.1, 5.1 we have the following results.

Theorem 6.1: If a generalized topological space (X, μ) is μ-$h\alpha$-T_1 then it is μ-$h\alpha$-T_0.

Theorem 6.2: If a generalized topological space (X, μ) is μ-$h\alpha$-D_1 then it is μ-$h\alpha$-D_0.

From the proposition 4.10 and theorem 6.1 we obtain the following result.

Theorem 6.3: If a generalized topological space (X, μ) is μ-$h\alpha$-T_1 then it is μ-$h\alpha$-D_0.

Remark 6.4: But the converse of above results is not true. We provide a counter example to justify it.

Example 6.5: Let $X = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, c, d\}\}$

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $M_{\mu_{h\alpha}} = \{a, b, c, d\}$

Collection of $D_{\mu_{h\alpha}}$-sets$= \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c\},$

$\{c\}, \{b, d\}, \{a, b\}, \{b\}\}$.

Here we observe that X is μ-$h\alpha$-T_0, μ-$h\alpha$-D_0, μ-$h\alpha$-D_1but not μ-$h\alpha$-T_1.

The connection between the aforementioned relations is summarized in the diagram below.

$$\begin{tikzcd}
\mu - h\alpha - T_0 \arrow[r, shift left] \arrow[d, shift right, "/" marking] & \mu - h\alpha - D_0 \arrow[l, shift left] \\
\mu - h\alpha - T_1 \arrow[u, shift right] \arrow[r, shift right] & \mu - h\alpha - D_1 \arrow[l, shift right, "/" marking] \arrow[ul] \arrow[u]
\end{tikzcd}$$

$P \longrightarrow Q$ means P implies Q	
$P \not\longrightarrow Q$ means P does not implies Q	

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