Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 1 (2024)

On Subspaces in Generalized Topological Spaces

Dr. S. B. Tadam

Department of Mathematics,

Shri R.L.T. College of Science, Akola, Maharashtra

Abstract: In this paper we present subspace of generalized topological space, μ-α-Subspace of a μ-α-generalized topological space and μ-$h\alpha$-Subspace of a μ-$h\alpha$-generalized topological space. Also we define interior and closure operators on these subspaces. Their properties, interrelationships, characterizations are obtained.

Keywords: Generalized topological space, μ-α-generalized topological space, μ-$h\alpha$-generalized topological space, μ-α-Subspace in generalized topological space, μ-$h\alpha$-Subspace in generalized topological space.

1. Introduction and Preliminaries :

The concept of subspaces in a topological space allows us to define a topology on a subset of a topological space that is strongly consistent with the topology on the larger space. The notion of subspace in topological spaces is extended to generalized topological spaces in this study. In 1963 in order to generalize the topology, N. Levein replaced open sets with semi-open sets in his paper "Semi-open sets and semi-continuity in topological spaces."α-open set, h-open set, and hα-
open set are concepts that were first presented by O. Najastad [15], A. Fadhil
[4], B.S. Abdullah, Sabih W. Askandar, and Ruquyah N. Balo [8],respectively.
A. Csaszar introduced the idea of generalized topological space in 2002. A family of subsets of a set that is closed in an arbitrary union but not in the condition of intersection is known as a generalized topology. Let X be a non empty set and $\mathcal{P}(X)$ be the power set of X. A subfamily μ of $\mathcal{P}(X)$ is called a generalized topology (GT, for short) on X if μ is closed under arbitrary union. (X,μ) is called a generalized topological space (GTS) [2].The members of μ are called μ-open sets and their complement are called μ-closed sets. In GTS(X,μ), here $M_\mu = \cup\{U: U \in \mu\}$. A GTS (X,μ) is called strong if $M_\mu = X$ [7]. Dr.S.B.Tadam and Ms. K.R.Sharma introduced the idea of μ- hα-generalized topological space[11].A subset A of a generalized topological space X is said μ- hα-open set denoted by (μ- hα-os) if for each set that is not empty U in X, $U \neq X$ and U is μ- α-open such that $A \subseteq i_\mu(A \cup U)$. The collection of all μ- hα-open set is denoted by $\mu_{h\alpha}$. i.e. $\mu_{h\alpha} = \{A : A$ is μ-hα-open set in $X\}$. Here $M_{\mu_{h\alpha}} = \cup\{U: U \in \mu_{h\alpha}\}$.

In this paper we present subspace of generalized topological space, μ-α-Subspace of a μ-α-generalized topological space and μ-$h\alpha$-Subspace of a μ-$h\alpha$-generalized topological space. Also we define interior and closure operators on these subspaces. Their properties, interrelationships, characterizations are obtained.
Here in this introduction, some definitions and basic concepts in topological space and generalized topological space have been given.

Definition 1.1: Let X^* be a subset of a topological space $(X,\mathcal{T})$. Then the relative topology for X^* is the collection $\mathcal{T}^*$ of all sets which are the intersections of X^* with the members of $\mathcal{T}$. i.e. $\mathcal{T}^* = \{G^* : G^* = G \cap X^*, G \in \mathcal{T}\}$.
Here $(X^*,\mathcal{T}^*)$ is called the subspace of a topological space $(X,\mathcal{T})$.

Definition 1.2: A subset A of a topological space $(X,\mathcal{T})$ is said to be
1. α-open set denoted by (α-os) [15] if $A \subseteq i(c(i(A))$

2.h-open set denoted by (h-os) [4] if for each set that is not empty U in X, $U \neq X$ and $U \in \mathcal{T}$, as a result $A \subseteq i(A \cup U)$.

Definition 1.3: [8] A subset A of a topological space $(X, \mathcal{T})$ is said to be $h\alpha$-open set denoted by ($h\alpha$-os) if for each set that is not empty U in X, $U \neq X$ and U is (α-os), as a result $A \subseteq i(A \cup U)$.
The complement of the $h\alpha$-open set is named $h\alpha$-closed set denoted by ($h\alpha$-cs).

Definition 1.4: Let (X, μ) be a GTS and A be a non empty subset of X.
1. μ-interior of A: [10] The interior of a subset A of X is $i_\mu(A) = \cup \{G_\mu : G_\mu \in \mu, G_\mu \subseteq A\}$
i.e. union of all μ-open set contained in A.
2. μ-closure of A: [10] The μ-closure of a subset A of X is $c_\mu(A) = \cap \{F_\mu : F_\mu\text{-}\mu\text{-closed set}, A \subseteq F_\mu\}$ i.e. intersection of all μ-closed set containing A.

Remark 1.5: Here μ-interior and μ -closure operators on a GTS (X, μ) satisfy the following properties:
1. $i_\mu(A) \subseteq A \subseteq c_\mu(A)$, for all $A \subseteq X$.
2. If $A \subseteq B \subseteq X$ then $i_\mu(A) \subseteq i_\mu(B)$ and $c_\mu(A) \subseteq c_\mu(B)$.

2. Subspace of a generalized topological space X:

Definition 2.1:[6] Let (X, μ) be a generalized topological space and X^* be any non empty subset of X. Then the relative generalized topology for X^* is the collection μ^* defined as $\mu^* = \{A^* : A^* = A \cap X^*, A \in \mu\}$. Hence $M_\mu^* = \bigcup\{A^* : A^* \in \mu^*\}$. Here (X^*, μ^*) is called the subspace of a generalized topological space (X, μ) and the members of μ^* are said to be the μ^*-open sets of (X^*, μ^*).

The set which is not μ^*-open is called μ^*-closed set of (X^*, μ^*). We denote the collection of all μ^*-closed set of X^* by $\mathcal{F}^*$. i.e. $\mathcal{F}^* = \{F^* : F^* \text{ is } \mu^*\text{-closed in } X^*\}$.

Theorem 2.2:[6] Subspace of a generalized topological space is also a generalized topological space.

Remark 2.3: From above theorem we observed that, arbitrary union of μ^*-open sets is again μ^*-open. Applying Demorgan's law we get the following result.

In a generalized topological space, arbitrary intersection of μ^*-closed sets is again μ^*-closed.

Definition 2.4: μ^*-interior of a set in X^*: The μ^*-interior of a subset A in X^* is the union of all μ^*-open sets contained in A and it is denoted by $i_\mu^*(A)$.

i.e. $i_\mu^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu, G^* \subseteq A\}$.

Here we observed that, $i_\mu^*(A)$ is the largest μ^*-open set contained in A.

Proposition 2.5: A set A is μ^*-open if and only if $A = i_\mu^*(A)$.

Proof: Suppose A is μ^*-open set in a subspace $X^*$of a generalized topological space X.

By definition, $i_\mu^*(A) = \cup \{G : G \in \mu^*, G \subseteq A\}$, arbitrary union of μ^*-open sets contained in A. Hence, $i_\mu^*(A)$ is the largest μ^*-open set contained in A.

Thus $A \subseteq i_\mu^*(A) \subseteq A. \Rightarrow A = i_\mu^*(A)$.

Conversely: Suppose $A = i_\mu^*(A)$. But $i_\mu^*(A)$ is the largest μ^*-open set contained in A.

$\Rightarrow A$ is μ^*-open set.

Definition 2.6: μ^*-closure of a set in X^*: The μ^*-closure of a subset A in X^* is the intersection of all μ^*-closed sets containing A and it is denoted by $c_\mu^*(A)$.

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 1 (2024)

i.e. $c_\mu^*(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}, A \subseteq F^*\}$, where $\mathcal{F} = C\mu$ = set of all μ-closed sets in X.

Here, $c_\mu^*(A)$ is the smallest μ^*-closed set containing A.

Proposition 2.7: A set E is μ^*-closed if and only if $E = c_\mu^*(E)$.

Proof: Suppose E is μ^*-closed set in a subspace $X^*$of a generalized topological space X.

By definition, $c_\mu^*(E) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}, E \subseteq F^*\}$, arbitrary intersection of μ^*-closed sets containing E. Hence,$c_\mu^*(E)$ is the smallest μ^*-closed set containing E.

Thus $E \subseteq c_\mu^*(E) \subseteq E. \Rightarrow E = c_\mu^*(E)$.

Conversely: Suppose $E = c_\mu^*(E)$. But $c_\mu^*(E)$ is the smallest μ^*-close set containing E.

$\Rightarrow E$ is μ^*-closed set.

Relationships 2.8:

1. From definition of μ^*-interior and μ^*-closure of a set A in X^*, it is observed that $i_\mu^*(A) \subseteq A \subseteq c_\mu^*(A)$.
2. $X^* \cap i_\mu(A) = X^* \cap \{\cup \{G : G \in \mu, G \subseteq A\}\}$
   $= \cup \{X^* \cap G : G \in \mu, X^* \cap G \subseteq A\}$
   $= i_\mu^*(A)$

   i.e. $i_\mu^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu, G^* \subseteq A\} = X^* \cap i_\mu(A)$.
3. $c_\mu^*(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}, A \subseteq F^*\} = X^* \cap c_\mu(A)$.

3. μ-α-Subspace of a μ-α-generalized topological space X:

Definition 3.1: [9] A subset A of a generalized topological space(X, μ) is:

1. μ-semiopen set in X if $A \subset c_\mu(i_\mu(A))$.
2. μ-α-open set if $A \subset i_\mu(c_\mu(i_\mu(A)))$.
3. μ-β-open set if $A \subset c_\mu(i_\mu(c_\mu(A)))$.

Theorem 3.2: (X, μ_α) is a generalized topological space.

Proof: Let (X, μ) be a generalized topological space and $\mu_\alpha = \{A \subseteq X : A \subseteq i_\mu(c_\mu(i_\mu(A)))\}$. As $\emptyset = i_\mu\left(c_\mu\left(i_\mu(\emptyset)\right)\right)$, hence $\emptyset \in \mu_\alpha$.

Now we have to show that μ_α is closed under arbitrary union.

Let $\{A_i\}$ be the arbitrary collection of μ_α-open sets in X.

i.e. for each $i, A_i \in \mu_\alpha \Rightarrow A_i \subseteq i_\mu(c_\mu(i_\mu(A_i)))$

For each i, $A_i \subseteq \cup_i A_i$

$\Rightarrow i_\mu(A_i) \subseteq i_\mu(\cup_i A_i)$

$\Rightarrow c_\mu(i_\mu(A_i)) \subseteq c_\mu(i_\mu(\cup_i A_i))$

$\Rightarrow i_\mu(c_\mu(i_\mu(A_i))) \subseteq i_\mu(c_\mu(i_\mu(\cup_i A_i)))$.

i.e. for $i, A_i \subseteq i_\mu(c_\mu(i_\mu(A_i))) \subseteq i_\mu(c_\mu(i_\mu \cup_i A_i)$

$\Rightarrow \cup_i A_i \subseteq i_\mu(c_\mu(i_\mu \cup_i A_i)$.

$\Rightarrow \bigcup_i A_i \in \mu_\alpha$.

Thus (X, μ_α) is a generalized topological space.

Proposition 3.3: A set A is μ_α-open if and only if $A = i_{\mu_\alpha}(A)$.

Proof: Suppose A is μ_α-open set in a generalized topological space X.

By definition, $i_{\mu_\alpha}(A) = \cup \{G : G \in \mu_\alpha, G \subseteq A\}$, arbitrary union of μ_α-open sets contained in A. Hence by above theorem, $i_{\mu_\alpha}(A)$ is the largest μ_α-open set contained in A.

Thus $A \subseteq i_{\mu_\alpha}(A) \subseteq A. \Rightarrow A = i_{\mu_\alpha}(A)$.

Conversely: Suppose $A = i_{\mu_\alpha}(A)$. But $i_{\mu_\alpha}(A)$ is the largest μ_α-open set contained in A.

$\Rightarrow A$ is μ_α-open set.

Definition3.4: μ-α-subspace of a generalized topological space X: Let (X, μ) be a generalized topological space and X^* be any non empty subset of X. Then μ-α-relative generalized topology for X^* is the collection μ_α^* defined as $\mu_\alpha^* = \{A^* : A^* = A \cap X^*, A \in \mu_\alpha\}$. Hence $M_{\mu_\alpha}^* = \cup\{A^* : A^* \in \mu_\alpha^*\}$.

Here (X^*, μ_α^*) is called the μ-α-subspace of a generalized topological space (X, μ_α) and the members of μ_α^* are said to be the μ_α^*-open sets of (X^*, μ_α^*).

The set which is not μ_α^*-open is called μ_α^*-closed set of (X^*, μ_α^*). We denote the collection of all μ_α^*-closed set of X^* by $\mathcal{F}_\alpha{}^*$. i.e. $\mathcal{F}_\alpha{}^* = \{F^* : F^*$ is μ_α^*-closed in $X^*\}$.

Theorem 3.5: (X^*, μ_α^*) is also a generalized topological space.

Proof: Let (X, μ) be a generalized topological space and (X^*, μ_α^*) be the μ-α-subspace of a generalized topological space (X, μ_α). By definition, $\mu_\alpha^* = \{A^* : A^* = A \cap X^*, A \in \mu_\alpha\}$

As $\emptyset \in \mu_\alpha \Rightarrow \emptyset = \emptyset \cap X^* \in \mu_\alpha^*$.

Thus we have only to show that μ_α^* is closed under arbitrary union.

Let $\{A_i{}^*\}$ be the arbitrary collection of μ_α^*-open sets in X^*.

i.e. for each $i, A_i{}^* \in \mu_\alpha^* \Rightarrow A_i{}^* = A_i \cap X^*$, for some $A_i \in \mu_\alpha$.

Thus we get an arbitrary collection $\{A_i\}$ of members of μ_α and μ_α is a generalized topology for X and hence $\bigcup_i A_i \in \mu_\alpha$.

$\therefore \bigcup_i A_i{}^* = \bigcup_i (A_i \cap X^*) = (\bigcup_i A_i) \cap X^* \in \mu_\alpha^*$.

Hence (X^*, μ_α^*) is also a generalized topological space.

Remark 3.6: In above theorem we observed that, arbitrary union of μ_α^*-open sets is again

μ_α^*-open. Hence by applying Demorgan's law we get, arbitrary intersection of μ_α^*-closed sets is again μ_α^*-closed.

Definition 3.7: μ_α^*-interior of a set in X^*: The μ_α^*-interior of a subset A in X^* is the union of all μ_α^*-open sets contained in A and it is denoted by $i_{\mu_\alpha}^*(A)$.

i.e. $i_{\mu_\alpha}^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu_\alpha, G^* \subseteq A\}$.

Proposition 3.8: A set A is μ_α^*-open if and only if $A = i_{\mu_\alpha}^*(A)$.

Proof: Suppose A is μ_α^*-open set in a μ-α-subspace $X^*$of a generalized topological space X. By definition, $i_{\mu_\alpha}^*(A) = \cup \{G : G \in \mu_\alpha^*, G \subseteq A\}$. Here, $i_{\mu_\alpha}^*(A)$ is the arbitrary union of μ_α^*-open sets and hence μ_α^*-open. Thus $i_{\mu_\alpha}^*(A)$ is the largest μ_α^*-open set contained in A.

Thus $A \subseteq i_{\mu_\alpha}^*(A) \subseteq A$.

$\Rightarrow A = i^*_{\mu_\alpha}(A)$.

Conversely: Suppose $A = i^*_{\mu_\alpha}(A)$, the largest μ^*_α-open set contained in A. Hence, A is μ^*_α-open set.

Definition 3.9: μ^*_α-closure of a set in X^*: The μ^*_α-closure of a subset A in X^* is the intersection of all μ^*_α-closed sets containing A and it is denoted by $c^*_{\mu_\alpha}(A)$.

i.e. $c^*_{\mu_\alpha}(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_\alpha, A \subseteq F^*\}$, where $\mathcal{F}_\alpha = C\mu_\alpha$ = set of all μ_α-closed sets in X.

Proposition 3.10: A set E is μ^*_α-closed if and only if $E = c^*_{\mu_\alpha}(E)$.

Proof: Suppose E is μ^*_α-closed set in a μ-α-subspace $X^*$of a generalized topological space X. By definition, $c^*_{\mu_\alpha}(E) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_\alpha, E \subseteq F^*\}$, arbitrary intersection of μ^*-closed sets containing E. Hence,$c^*_\mu(E)$ is the smallest μ^*-closed set containing E.

Thus $E \subseteq c^*_\mu(E) \subseteq E. \Rightarrow E = c^*_\mu(E)$.

Conversely: Suppose $E = c^*_{\mu_\alpha}(E)$. But $c^*_{\mu_\alpha}(E)$ is the smallest μ^*_α-closed set containing E.

$\Rightarrow E$ is μ^*_α-closed set.

Relationships 3.11:

1. The relation between μ^*_α-interior and μ^*_α-closure of a set in $X^*$is $i^*_{\mu_\alpha}(A) \subseteq A \subseteq c^*_{\mu_\alpha}(A)$.

2. The relation between μ^*_α-interior and μ_α-interior of a set in a generalized topological space is $i^*_{\mu_\alpha}(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu_\alpha, G^* \subseteq A\} = X^* \cap i_{\mu_\alpha}(A)$.

3. The relation between μ^*_α-closure and μ_α-closure of a set in a generalized topological space is $c^*_{\mu_\alpha}(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_\alpha, A \subseteq F^*\} = X^* \cap c_{\mu_\alpha}(A)$.

4. μ-$h\alpha$-subspace of a μ-$h\alpha$-generalized topological space X:

Definition 4.1: μ-$h\alpha$-subspace of a μ-$h\alpha$-generalized topological space X**:** Let (X, μ) be a generalized topological space and X^* be any non empty subset of X. Then μ-$h\alpha$-relative generalized topology for X^* is the collection $\mu^*_{h\alpha}$ defined as $\mu^*_{h\alpha} = \{A^* : A^* = A \cap X^*, A \in \mu_{h\alpha}\}$. Hence $M^*_{\mu_{h\alpha}} = \cup\{A^* : A^* \in \mu^*_{h\alpha}\}$.

Here $(X^*, \mu^*_{h\alpha})$ is called the μ-$h\alpha$-subspace of a μ-$h\alpha$-generalized topological space $(X, \mu_{h\alpha})$ and the members of $\mu^*_{h\alpha}$ are said to be the $\mu^*_{h\alpha}$-open sets of $(X^*, \mu^*_{h\alpha})$.

The set which is not $\mu^*_{h\alpha}$-open is called $\mu^*_{h\alpha}$-closed set of $(X^*, \mu^*_{h\alpha})$. We denote the collection of all $\mu^*_{h\alpha}$-closed set of X^* by $\mathcal{F}_{h\alpha}{}^*$. i.e.$\mathcal{F}_{h\alpha}{}^* = \{F^* : F^*$ is $\mu^*_{h\alpha}$-closed in $X^*\}$.

Theorem 4.2: $(X^*, \mu^*_{h\alpha})$ is also a generalized topological space.

Proof: Let (X, μ) be a generalized topological space and $(X^*, \mu^*_{h\alpha})$ be the μ-$h\alpha$-subspace of a μ-$h\alpha$-generalized topological space $(X, \mu_{h\alpha})$. By definition, $\mu^*_{h\alpha} = \{A^* : A^* = A \cap X^*, A \in \mu_{h\alpha}\}$.

As $\emptyset \in \mu_{h\alpha} \Rightarrow \emptyset = \emptyset \cap X^* \in \mu^*_{h\alpha}$.

Thus we have only to show that $\mu^*_{h\alpha}$ is closed under arbitrary union.

Let $\{G_i{}^*\}$ be the arbitrary collection of $\mu^*_{h\alpha}$-open sets in X^*.

i.e. for each i, $G_i{}^* \in \mu^*_{h\alpha} \Rightarrow G_i{}^* = G_i \cap X^*$, for some $G_i \in \mu_{h\alpha}$.

Thus we get an arbitrary collection $\{G_i\}$ of members of $\mu_{h\alpha}$ and $\mu_{h\alpha}$ is a generalized topology for X and hence $\cup_i G_i \in \mu_{h\alpha}$.

$\Rightarrow (\cup_i G_i) \cap X^* \in \mu^*_{h\alpha}$.

$\therefore \cup_i G_i{}^* = \cup_i(G_i \cap X^*) = (\cup_i G_i) \cap X^* \in \mu^*_{h\alpha}$.

Hence $(X^*, \mu_{h\alpha}^*)$ is also a generalized topological space.

Definition 4.3: $\mu_{h\alpha}^*$-interior of a set in X^*: The $\mu_{h\alpha}^*$-interior of a subset A in X^* is the union of all $\mu_{h\alpha}^*$-open sets contained in A and it is denoted by $i_{\mu_{h\alpha}}^*(A)$.

i.e. $i_{\mu_{h\alpha}}^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu_{h\alpha}, G^* \subseteq A\}$.

Proposition 4.4: A set A is $\mu_{h\alpha}^*$-open if and only if $A = i_{\mu_{h\alpha}}^*(A)$.

Proof: Suppose A is $\mu_{h\alpha}^*$-open set in a μ-$h\alpha$-subspace X^*.

By definition, $i_{\mu_{h\alpha}}^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu_{h\alpha}, G^* \subseteq A\}$, arbitrary union of $\mu_{h\alpha}^*$-open sets contained in A. Hence, $i_{\mu_{h\alpha}}^*(A)$ is the largest $\mu_{h\alpha}^*$-open set contained in A.

Thus $A \subseteq i_{\mu_{h\alpha}}^*(A) \subseteq A. \Rightarrow A = i_{\mu_{h\alpha}}^*(A)$.

Conversely: Suppose $A = i_{\mu_{h\alpha}}^*(A)$. But $i_{\mu_{h\alpha}}^*(A)$ is the largest $\mu_{h\alpha}^*$-open set contained in A.

$\Rightarrow A$ is $\mu_{h\alpha}^*$-open set.

Definition 4.5: $\mu_{h\alpha}^*$-closure of a set in X^*: The $\mu_{h\alpha}^*$-closure of a subset A in X^* is the intersection of all $\mu_{h\alpha}^*$-closed sets containing A and it is denoted by $c_{\mu_{h\alpha}}^*(A)$.

i.e. $c_{\mu_{h\alpha}}^*(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_{h\alpha}, A \subseteq F^*\}$, where $\mathcal{F}_{h\alpha} = C\mu_{h\alpha}$ = set of all $\mu_{h\alpha}$-closed sets in X.

Proposition 4.6: A set E is $\mu_{h\alpha}^*$-closed if and only if $E = c_{\mu_{h\alpha}}^*(E)$.

Proof: Suppose E is $\mu_{h\alpha}^*$-closed set in a μ-$h\alpha$-subspace $X^*$of a generalized topological space X. By definition, $c_{\mu_{h\alpha}}^*(E) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_{h\alpha}, E \subseteq F^*\}$, arbitrary intersection of $\mu_{h\alpha}^*$-closed sets containing E. Hence,$c_{\mu_{h\alpha}}^*(E)$ is the smallest $\mu_{h\alpha}^*$-closed set containing E. Thus $E \subseteq c_{\mu_{h\alpha}}^*(E) \subseteq E. \Rightarrow E = c_{\mu_{h\alpha}}^*(E)$.

Conversely: Suppose $E = c_{\mu_{h\alpha}}^*(E)$. But $c_{\mu_{h\alpha}}^*(E)$ is the smallest $\mu_{h\alpha}^*$-closed set containing E. Hence,E is $\mu_{h\alpha}^*$-closed set.

Relationships 4.7:

1. Relation between $M_{\mu_{h\alpha}}^*$ and $M_{\mu_{h\alpha}}$: We know $M_{\mu_{h\alpha}} = \cup \{A : A \in \mu_{h\alpha}\}$.
For any $x \in M_{\mu_{h\alpha}}^*$ there exists $A^* \in \mu_{h\alpha}^*$ such that $x \in A^* \subseteq M_{\mu_{h\alpha}}^*$.
But $A^* \in \mu_{h\alpha}^* \Rightarrow A^* = A \cap X^*, A \in \mu_{h\alpha}$.
Now $\in \mu_{h\alpha} \Rightarrow A \subseteq M_{\mu_{h\alpha}}$.
Thus $x \in A^* \subseteq A \subseteq M_{\mu_{h\alpha}}$
$\Rightarrow x \in M_{\mu_{h\alpha}}$.
$\Rightarrow M_{\mu_{h\alpha}}^* \subseteq M_{\mu_{h\alpha}}$.
2. The relation between $\mu_{h\alpha}^*$-interior and $\mu_{h\alpha}^*$-closure of a set in generalized topological space is $i_{\mu_{h\alpha}}^*(A) \subseteq A \subseteq c_{\mu_{h\alpha}}^*(A)$.
3. The relation between $\mu_{h\alpha}^*$-interior and $\mu_{h\alpha}$-interior of a set in generalized topological space is $i_{\mu_{h\alpha}}^*(A) = \cup \{G^* : G^* = G \cap X^*, G \in \mu_{h\alpha}, G^* \subseteq A\} = X^* \cap i_{\mu_{h\alpha}}(A)$.
4. The relation between $\mu_{h\alpha}^*$-closure and $\mu_{h\alpha}$-closure of a set in generalized topological space is $c_{\mu_{h\alpha}}^*(A) = \cap \{F^* : F^* = F \cap X^*, F \in \mathcal{F}_{h\alpha}, A \subseteq F^*\} = X^* \cap c_{\mu_{h\alpha}}(A)$.

References:

[1] Csaszar, Generalized Open sets, Acta Math. Hungar., 75(1997), 65-87.

[2] Csaszar, Generalized topology, generalized continuity.(2002). Acta Mathematica Hungarica, 96 (4). 351-357. http://d.doi.org/10.1023/A:1019713018007

[3] Csaszar, δ- and θ- modification of generalized topologies, Acta Math. Hungar., 120 (3) (2008), 275-279.

[4] Fadhil, "On h-open Sets and h-Continuous Function", J. Appl., Compt. Math., vol. 9, pp. 1-5, 2020.

[5] B.K.Tyagi and R. Choudhary, On Generalized Closure Operator in Generalized Topological Spaces, International Journal of Computer Applications, Vol. 82 (15) (2013), 1-5.

[6] B.K.Tyagi, Harsh V.S. Chauhan, On generalized closed sets in generalized topological spaces, CUBO A Mathematical Journal, Vol. 18, N0-01,(27-45), December 2016.

[7] K. Tyagi and H.V. S. Chauhan, On some separation axioms in generalized topological spaces (2018), Symposium of general topology.

[8] Beyda S. Abdullah, Sabih W. Askandar, Ruquayah N. Balo, $h\alpha$ -Open Sets in Topological Spaces, Jounral of education and science (ISSN 1812-125X) , Vol:31,No:03,2022(91-98).

[9] Dhanabalan, AP and Santhi C , A Class of Separation Axioms in Generalized Topology, Mathematical Jounral of interdisciplinary sciences Vol.4 No.2 (2016) PP. 151-159

[10] Dipankar Dey, Dhananjay Mandal and Manabendra Nath Mukherjee, Uniformity on generalized topological spaces, Arab Journal of Mathematical Sciences Emerald Publishing Limited(e-ISSN:2588-9214, p-ISSN: 1319-5166),DOI 10.1108/AJMS-03-2021-0058

[11] Dr.S.B.Tadam, Ms.K.R.Sharma, On μ-$h\alpha$-Open sets and μ-$h\alpha$-Closed sets in Generalized Topological Spaces, Tuijin Jishu/Journal of Propulsion Technology,vol.44 No.6(2023),pp. 279-287.

[12] Dunya M. Hammed, On Arps-Closed sets in Topological Spaces, Eng. & Tech. Journal, Vol.32,Part(B), No.2, 2014

[13] J. L. Kelley, General Topology, Van Nostrand, Princeton, New Jersey, 1955.

[14] M. Jeyaraman,S.C.Vasthirani, O. Ravi and R. Muthuraj, On α^*-sets in Generalized Topological Spaces, International Journal of Current Research in Science and Technology(ISSN: 2394-5745), Vol. 1, Issue 5 (2015), 39-43.

[15] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[16] O. Njastad, On Some Classes of Nearly Open Sets, Pacific J. Math., vol. 15, no. 3, pp. 961-970, 1965. DOI: http://dx.doi.org/10.2140/pjm.1965.15.961

[17] Rasool Khayyeri, On Base for Generalized Topological Spaces, Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 48, 2377-2383.

[18] W. J. Pervin, Foundation of General Topology, Publisher: Academic Press.