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On μ-$h\alpha$-Open sets and μ-$h\alpha$-Closed sets in Generalized Topological Spaces

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Abstract: In this paper we present μ-h-Open sets, μ-$h\alpha$-Open sets, μ-$h\alpha$-Closed sets, μ-$h\alpha$-interior and μ-$h\alpha$-closure operator in generalized topological spaces(GTS). Their properties, interrelationships and several characterizations are obtained.

Keywords: *Generalized topological spaces, μ-h-Open sets, μ-$h\alpha$-Open sets, μ-$h\alpha$-neighborhood system, μ-$h\alpha$-limit points, μ-$h\alpha$-Closed sets, μ-$h\alpha$-interior and μ-$h\alpha$-closure operator in generalized topological spaces.*

1. Introduction and Preliminaries:

A concept of α-open set in topological space was initiated by O. Najastad [14]. A Fadhil [4] introduced h-open set in topological spaces. Further this study extended to $h\alpha$-open set in topological spaces by B. S. Abdullah, Sabh W. Askandar, Ruqayah N. Balo [8]. The aim of this paper is to introduce and study the properties of μ-$h\alpha$-Open sets and μ-$h\alpha$-Closed sets in generalized topological spaces (GTS), introduced by A. Csaszar [2] which is not close in condition of intersection, specifically they are close in arbitrary union. Let X be a non empty set and $\mathcal{P}(X)$ be the power set of X. A subset μ of $\mathcal{P}(X)$ is called a generalized topology (GT, for short) on X if μ is closed under arbitrary union. (X,μ) is called a generalized topological space (GTS) [2].The members of μ are called μ-open sets and their complement are called μ-closed sets. In GTS(X,μ), we define $M_\mu = \cup\{U: U \in \mu\}$. A GTS (X,μ) is called strong if $M_\mu = X$ [7]. It is well known that generalized topology in the sense of Csaszar [2] is a generalization of topology on set.

In this paper, we look into the relationships among μ-open set, α-open set and μ-$h\alpha$ open sets in generalized topological space in section 2. Subsequently in section 3 we introduce and studied μ-$h\alpha$-interior and μ-$h\alpha$-neighborhoods and its properties. At last in the series we look at μ-$h\alpha$-limit point, μ-$h\alpha$ closed set, μ-$h\alpha$- closure operator and its properties in section 4.

Here in this introduction, some definitions and basic concepts in topological space and generalized topological space have been given.

In topological space $(X,\mathcal{T})$ we denote the interior (resp. closure) of a subset A of X by $i(A)$, $(c(A))$.

Definition 1.1: A subset A of a topological space $(X,\mathcal{T})$ is said to be

1. α-open set denoted by (α-os) [14] if $A \subseteq i(c(i(A))$

2.h-open set denoted by (h-os) [4] if for each set that is not empty U in X, $U \neq X$ and $U \in \mathcal{T}$, as a result $A \subseteq i(A \cup U)$.

Definition 1.2: [8] A subset A of a topological space $(X,\mathcal{T})$ is said to be $h\alpha$-open set denoted by ($h\alpha$-os) if for each set that is not empty U in X, $U \neq X$ and U is (α-os), as a result $A \subseteq i(A \cup U)$. The complement of the $h\alpha$-open set is named $h\alpha$-closed set denoted by ($h\alpha$-cs).

Lemma 1.3: [4] Each open set in a topological space is (h-os).

Lemma 1.4: [14] Each open set in a topological space is (α-os).

Lemma 1.5: [8] Each (h-os) in a topological space is ($h\alpha$-os).

Lemma 1.6: [8] Any open set in a topological space is ($h\alpha$-os).

Definition 1.7: Let (X, μ) be a GTS and A be a non empty subset of X.

1. μ-interior of A: [10] The interior of a subset A of X is $i_\mu(A) = \cup \{G_\mu : G_\mu \in \mu, G_\mu \subseteq A\}$

i.e. union of all μ-open set contained in A.

2. μ-closure of A: [10] The μ-closure of a subset A of X is $c_\mu(A) = \cap \{F_\mu : F_\mu\text{-}\mu\text{-closed set}, A \subseteq F_\mu\}$ i.e. intersection of all μ-closed set containing A.

Remark 1.8: Here μ-interior and μ -closure operators on a GTS (X, μ) satisfy the following properties:

1. $i_\mu(A) \subseteq A \subseteq c_\mu(A)$, for all $A \subseteq X$.

2. If $A \subseteq B \subseteq X$ then $i_\mu(A) \subseteq i_\mu(B)$ and $c_\mu(A) \subseteq c_\mu(B)$.

Definition 1.9: [9] A subset A of a generalized topological space(X, μ) is:

1. μ-semiopen set in X if $A \subset c_\mu(i_\mu(A))$.

2. μ-α-open set if $A \subset i_\mu(c_\mu(i_\mu(A)))$.

3. μ-β-open set if $A \subset c_\mu(i_\mu(c_\mu(A)))$.

2. μ-$h\alpha$- Open Sets in Generalized Topological Spaces:

Definition 2.1: A subset A of a generalized topological space X is μ-h-open set if for each $U \neq \emptyset$ in X, $U \neq X$and $U \in \mu$, $A \subset i_\mu(A \cup U)$. We denote the collection of all (μ-h-os) in GTS X by μ_h. i.e. $\mu_h = \{A : A \text{ is } \mu\text{-}h\text{-open set in } X\}$.

Definition 2.2: A subset A of GTS X is said to be μ-$h\alpha$-open set denoted by (μ-$h\alpha$-os) if for each set that is not empty U in X, $U \neq X$ and U is μ-α open such that $A \subseteq i_\mu(A \cup U)$. We denote the collection of all (μ-$h\alpha$-os) in GTS X by $\mu_{h\alpha}$. i.e. $\mu_{h\alpha} = \{A : A \text{ is } \mu\text{-}h\alpha\text{-open set in } X\}$.

We define $M_{\mu_{h\alpha}} = \cup \{A : A \in \mu_{h\alpha}\}$.

The set which is not μ-$h\alpha$-open is called μ-$h\alpha$-closed set denoted by (μ-$h\alpha$-cs). We denote the collection of all (μ-$h\alpha$-cs) in GTS X by $\mathcal{F}_{\mu_{h\alpha}}$. i.e. $\mathcal{F}_{\mu_{h\alpha}} = \{F : F \text{ is } \mu\text{-}h\alpha\text{-closed set in } X\}$.

Proposition 2.3: Each μ-open set in GTS is μ-h- open set.

Proof: Let (X, μ) be a GTS and A be an μ-open set in X.

To show that A is μ-h-open set, we have to show for each $U \neq \emptyset$ in X, $U \neq X$and $U \in \mu$, $A \subset i_\mu(A \cup U)$.

We know $A \subseteq A \cup U$, for each set $U \in \mu$

$\Rightarrow i_\mu(A) \subseteq i_\mu(A \cup U)$

$\Rightarrow A \subseteq i_\mu(A \cup U) \quad (\because A \in \mu, A = i_\mu(A))$

Hence each μ-open set in GTS X is μ-h-open set.

Remark 2.4: The converse of the above proposition is not true. We justify it through the following example.

Example 2.5: Let $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a\}, \{a, c\}\}$, $\mu_h = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Here we observe that, each μ-open set is μ-h- open set but $\{c\}$ is μ-h-open set which is not a μ-open set.

Proposition 2.6: Each μ-open set in GTS X is μ-α-open set.

Proof: Let (X, μ) be a GTS and A be an μ-open set in X.

To show that A is μ-α-open set, we have to show $A \subseteq i_\mu(c_\mu(i_\mu(A)))$

Since $A \in \mu$, $A = i_\mu(A)$
....................(1)

Also $A \subseteq c_\mu(A) = c_\mu(i_\mu(A))$

i.e. $A \subseteq c_\mu(i_\mu(A))$

$\therefore i_\mu(A) \subseteq i_\mu(c_\mu\left(i_\mu(A)\right))$

$\Rightarrow A \subseteq i_\mu(c_\mu\left(i_\mu(A)\right))$ (from(1))

$\Rightarrow A$ is μ-α-open set.

Hence each μ-open set in GTS X is μ-α-open set.

Proposition 2.7: Each μ-h- open set in GTS X is μ-$h\alpha$-open set.

Proof: Let (X, μ) be a GTS and A be an μ-h-open set in X. Thus for each $U \neq \emptyset$ in X, $U \neq X$and $U \in \mu$, we have $A \subset i_\mu(A \cup U)$. Since by Proposition 2.6, each μ-open set in GTS X is μ-α-open set, $U \in \mu_\alpha$.Thus for each set $U \neq \emptyset$ in X, $U \neq X$ and $U \in \mu_\alpha$, we have $A \subset i_\mu(A \cup U)$.

$\Rightarrow A$ is μ-$h\alpha$-open set. Hence each μ-h- open set in GTS X is μ-$h\alpha$-open set.

Remark 2.8: From Proposition 2.3 and 2.7 we get, each μ-open set in GTS X is μ-$h\alpha$-open set and hence $M_\mu \subseteq M_{\mu_{h\alpha}}$. But the converse of the proposition 2.6 and 2.7 are not true. We justify it by giving a counter example.

Example 2.9: Let $X = \{a, b, c, d, e\}$ **,** $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$,

$\mu_\alpha = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$,

$\mu_h = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$.

Here we observe that, each μ-open set is μ-α-open set but $\{a, b\}$ is μ-α-open set which is not μ-open. i.e. the converse of the proposition 2.6 is not true. Also, each μ-h- open set is μ-$h\alpha$-open set but $\{b, c, d\}$ is a μ-$h\alpha$-open set which is not μ-h- open. Hence the converse of the proposition 2.7 is not true.

In this example we also observed that, $\{a, b\}$ is μ-α-open set which is not μ-$h\alpha$-open and $\{b, c, d\}$ is μ-$h\alpha$-open but not μ-α-open set. Hence there is no relationship between μ-$h\alpha$-open set and μ-α-open set.

Theorem 2.10: In a GTS X, arbitrary union of μ-$h\alpha$-open set is μ-$h\alpha$-open set.

Proof: Let (X, μ) be a GTS and $\{G_\lambda\}$ be any arbitrary collection of μ-$h\alpha$-open sets in X. To show that $\cup_\lambda G_\lambda$ is μ-$h\alpha$-open set in X we have to show that for each set that is not empty U in X, $U \neq X$ and U is μ-α open such that $\cup_\lambda G_\lambda \subseteq i_\mu((\cup_\lambda G_\lambda) \cup U)$.

Let U be any non empty proper subset of X, such that U is μ-α open in X then we have to show that

$\cup_\lambda G_\lambda \subseteq i_\mu((\cup_\lambda G_\lambda) \cup U)$.

Let $x \in \cup_\lambda G_\lambda \Rightarrow x \in G_\lambda$ for some λ ....(2)

As G_λ is μ-$h\alpha$-open set in $X \Rightarrow G_\lambda \subseteq i_\mu(G_\lambda \cup U)$

i.e. $G_\lambda \subseteq i_\mu(G_\lambda \cup U) \subseteq i_\mu((\cup_\lambda G_\lambda) \cup U)$

$\Rightarrow G_\lambda \subseteq i_\mu((\cup_\lambda G_\lambda) \cup U)$ ....(3)

From (2) and (3) we get, $x \in i_\mu((\cup_\lambda G_\lambda) \cup U)$ ....(4)

From (2) and (4) we get, $\cup_\lambda G_\lambda \subseteq i_\mu((\cup_\lambda G_\lambda) \cup U)$

Hence, $\cup_\lambda G_\lambda$ is μ-$h\alpha$-open sets in X.

Remark 2.11:

i] Applying DeMorgan's law to the above theorem we get the following result.

In a GTS X, arbitrary intersection of μ-$h\alpha$-closed set is μ-$h\alpha$-closed set.

ii] The following example shows that finite intersection of μ-$h\alpha$-open set need not be μ-$h\alpha$-Open set.

Example 2.12: Let $X = \{a, b, c, d, e\}$ **,** $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$,

$\mu_\alpha = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$ and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Here $\{a, b, c\}$ and $\{a, b, d\}$ are μ-$h\alpha$-open sets but their intersection$\{a, b, d\} \cap \{b, c, d\} = \{b, d\} \notin \mu_{h\alpha}$. Thus, finite intersection of μ-$h\alpha$-open set need not be μ-$h\alpha$-open set.

Remark 2.13:

From theorem 2.10 and example 2.12, it is observed that $(X, \mu_{h\alpha})$ is a generalized topological space but not a topological space.

3. μ-$h\alpha$-Interior and μ-$h\alpha$-Neighborhoods in Generalized Topological Space:

Definition 3.1: μ-$h\alpha$-interior of A: The μ-$h\alpha$-interior of a subset A of a GTS X is the union of all μ-$h\alpha$-open sets contained in A and will be denoted by $i_{\mu_{h\alpha}}(A)$.i.e. $i_{\mu_{h\alpha}}(A) = \cup \{G: G\text{-} \mu\text{-}h\alpha\text{-open set}, G \subseteq A\}$.

Remark 3.2: From theorem 2.10 we get, $i_{\mu_{h\alpha}}(A)$ is the largest μ-$h\alpha$-open sets contained in A.

Theorem 3.3: A subset A of a GTS X is μ-$h\alpha$-open set iff $A = i_{\mu_{h\alpha}}(A)$.

Proof: Let A be a μ-$h\alpha$-open subset of a GTS X.By definition, $i_{\mu_{h\alpha}}(A) = \cup \{G_{\mu_{h\alpha}}: G_{\mu_{h\alpha}}\text{-} \mu\text{-}h\alpha\text{-open set}, G_{\mu_{h\alpha}} \subseteq A\}$ is the largest μ-$h\alpha$-open sets contained in A.

i.e. $i_{\mu_{h\alpha}}(A) \subseteq A$ ....(5)

But A is μ-$h\alpha$-open set and $A \subseteq A$ hence $A \subseteq i_{\mu_{h\alpha}}(A)$ ....(6)

From (5) and (6) we get, $A = i_{\mu_{h\alpha}}(A)$.

Conversely: Let $A = i_{\mu_{h\alpha}}(A)$

As $i_{\mu_{h\alpha}}(A)$ is the largest μ-$h\alpha$-open set, hence A is a μ-$h\alpha$-open set.

Remark 3.4: μ-$h\alpha$-interior operators on a GTS X obey the following properties.

i] $i_{\mu_{h\alpha}}(i_{\mu_{h\alpha}}(A)) = i_{\mu_{h\alpha}}(A)$,

ii] If $A \subseteq B$ then $i_{\mu_{h\alpha}}(A) \subseteq i_{\mu_{h\alpha}}(B)$,

iii] $i_{\mu_{h\alpha}}(A \cap B) \subseteq i_{\mu_{h\alpha}}(A) \cap i_{\mu_{h\alpha}}(B)$,

iv] $i_{\mu_{h\alpha}}(A) \cup i_{\mu_{h\alpha}}(B) \subseteq i_{\mu_{h\alpha}}(A \cup B)$.

Definition 3.5: In a GTS X, μ-$h\alpha$-neighborhood of a point x is any set which contains μ-$h\alpha$-open set containing the point. i.e. E is μ-$h\alpha$-neighborhood of a point x if there is μ-$h\alpha$-open set G containing x such that $x \in G \subseteq E$.

Lemma 3.6: In a GTS X, E is μ-$h\alpha$-neighborhood of a point x iff $x \in i_{\mu_{h\alpha}}(E)$.

Proof: Let E be a μ-$h\alpha$-neighborhood of a point x.Then by definition, there is μ-$h\alpha$-open set H containing x such that $x \in H \subseteq E$.

$\Rightarrow x \in H \subseteq \cup \{G: G\text{-} \mu\text{-}h\alpha\text{-open set}, G \subseteq E\} = i_{\mu_{h\alpha}}(E)$

$\Rightarrow x \in i_{\mu_{h\alpha}}(E)$

i.e. if E is μ-$h\alpha$-neighborhood of a point x then $x \in i_{\mu_{h\alpha}}(E)$

Conversely: Let $x \in i_{\mu_{h\alpha}}(E)$. By remark 3.2, $i_{\mu_{h\alpha}}(E)$ is the largest μ-$h\alpha$-open set contained in E.

Thus $i_{\mu_{h\alpha}}(E)$ is the μ-$h\alpha$-open set such that $x \in i_{\mu_{h\alpha}}(E) \subseteq E$.

$\Rightarrow E$ is μ-$h\alpha$-neighborhood of a point x.

Theorem 3.7: In a GTS X, a set is μ-$h\alpha$-open iff it is a μ-$h\alpha$-neighborhood of each of its point.

Proof: Let X be a GTS and $G \subseteq X$. First suppose that G is μ-$h\alpha$-open. Then for any $x \in G$ we have,

$x \in G \subseteq G$ and hence G is μ-$h\alpha$-neighborhood of a point x. Thus G is a μ-$h\alpha$-neighborhood of each of its point.

Conversely: Suppose G is a μ-$h\alpha$-neighborhood of each of its points. Then for each $x \in G$, there is μ-$h\alpha$-open set H_x such that $x \in H_x \subseteq G$.Then $G = \bigcup_{x\in G} H_x$, arbitrary union of μ-$h\alpha$-open set. Hence by theorem 2.10, G is μ-$h\alpha$-open set.

Definition 3.8: Let X be a GTS and $x \in X$. Let $\mathcal{N}_x$ be the set of all μ-$h\alpha$-neighborhoods of x in X. Then the family $\mathcal{N}_x$ is called the μ-$h\alpha$-neighborhood system at x.

Theorem 3.9: Let X be a GTS and for $x \in X$, let $\mathcal{N}_x$ be the μ-$h\alpha$-neighborhood system at x. Then,

i] If $U \in \mathcal{N}_x$ then $x \in U$.

ii] If $V \in \mathcal{N}_x$ and $V \subseteq U$ then $U \in \mathcal{N}_x$.

iii] For any $U, V \in \mathcal{N}_x$, $U \cup V \in \mathcal{N}_x$.

iv] A set G is μ-$h\alpha$-open set in X iff $G \in \mathcal{N}_x$ for all $x \in G$.

v] If $U \in \mathcal{N}_x$ then there exists $V \in \mathcal{N}_x$ such that $V \subseteq U$, $V \in \mathcal{N}_y$ for all $y \in V$ and $U \in \mathcal{N}_y$ for all $y \in V$.

Proof: Properties [i] to [iv**]** are obvious from the definition of neighborhood of a point given in

definition 3.5. Now for [v], consider $U \in \mathcal{N}_x$ i.e. U is μ-$h\alpha$-neighborhood of a point x. Thus there exist μ-$h\alpha$-open set V such that $x \in V \subseteq U$. As V is μ-$h\alpha$-open set then by Theorem 3.7, V is μ-$h\alpha$-neighborhood of each of its points. i.e. $V \in \mathcal{N}_y$ for all $y \in V$. Also from [ii], $V \subseteq U \Rightarrow U \in \mathcal{N}_y$ for all $y \in V$.

Theorem 3.10: Let X be a set and suppose for each $x \in X$, a non-empty family $\mathcal{N}_x{}^*$ of subsets of X is given, satisfying [i], [ii] and [v] of the theorem 3.9 above. Let μ^* be the family of all subsets of X which are μ-$h\alpha$-neighborhood of each of their points; i.e. $\mu^* = \{G : x \in G$ implies that $G \in \mathcal{N}_x{}^*\}$. Then μ^* is a generalized topology for X, and if $\mathcal{N}_x$ is the collection of all μ-$h\alpha$-neighborhoods of x defined by the generalized topology μ^*, then $\mathcal{N}_x{}^*$=$\mathcal{N}_x$ for every $x \in X$.

Proof: To show that $\mu^* = \{G : x \in G$ implies that $G \in \mathcal{N}_x{}^*\}$ is a generalized topology for X, we have to show that $\emptyset \in \mu^*$ and $\bigcup_{\lambda\in I} G_\lambda \in \mu^*$ where $G_\lambda \in \mu^*$ for every $\lambda \in I \neq \emptyset$. Clearly, $\emptyset \in \mu^*$. Let $G_\lambda \in \mu^*$ for $\lambda \in I \neq \emptyset$ and $x \in \bigcup_{\lambda\in I} G_\lambda$. Now $x \in \bigcup_{\lambda\in I} G_\lambda \Rightarrow x \in G_\lambda$ for some $\lambda \in I$.By definition of μ^*, $x \in G_\lambda \Rightarrow G_\lambda \in \mathcal{N}_x{}^*$. As, $G_\lambda \in \mathcal{N}_x{}^*$ and $G_\lambda \subseteq \bigcup_{\lambda\in I} G_\lambda$ thus by theorem 3.9 [ii], $\bigcup_{\lambda\in I} G_\lambda \in \mathcal{N}_x{}^*$. i.e. $x \in \bigcup_{\lambda\in I} G_\lambda \Rightarrow \bigcup_{\lambda\in I} G_\lambda \in \mathcal{N}_x{}^*$. Hence $\bigcup_{\lambda\in I} G_\lambda \in \mu^*$.

($\mathcal{N}_x{}^*$=$\mathcal{N}_x$): If $N \in \mathcal{N}_x$, i.e. N is μ-$h\alpha$-neighborhoods of x then there exists $G \in \mu^*$ such that $x \in G \subseteq N$. From the definition of μ^*, $x \in G$ implies that $G \in \mathcal{N}_x{}^*$ and hence $N \in \mathcal{N}_x{}^*$ by [ii]. Thus $\mathcal{N}_x \subseteq \mathcal{N}_x{}^*$.

Now suppose that $N \in \mathcal{N}_x{}^*$. Let us define the set G as, $G = \{y : N$ is the μ-$h\alpha$-neighborhoods of $y\}$ to be all points which have N as a μ-$h\alpha$-neighborhood. Clearly,$x \in G$, while by [i], every point with N as a μ-$h\alpha$-neighborhood is in N, so $y \in N \; \forall \, y \in G$ implies that $G \subseteq N$. Now we will show that $G \in \mu^*$; i.e., for every $y \in G$ implies that $G \in \mathcal{N}_y{}^*$. Let $y \in G$, so that N is the μ-$h\alpha$-neighborhood of y i.e. $N \in \mathcal{N}_y{}^*$. By [v], there exists a set N^* such that $N^* \in \mathcal{N}_y{}^*$, and if $z \in N^*$ then $N \in \mathcal{N}_z{}^*$. The definition of G shows that $N \in \mathcal{N}_z{}^*$ implies that $z \in G$, hence, $N^* \subseteq G$ and, by [ii], $G \in \mathcal{N}_y{}^*$.

4. μ-$h\alpha$-Closed sets in Generalized Topological Spaces:

Definition 4.1: For a GTS X, a point $x \in M_{\mu_{h\alpha}}$ is a μ-$h\alpha$-limit point of a subset E iff every μ-$h\alpha$-open set containing x contains a point of E different from x; i.e. if $x \in G \in \mu_{h\alpha}$, then $E \cap G - \{x\} \neq \emptyset$. The set of all μ-$h\alpha$-limit points of a set E is called the μ-$h\alpha$-derived set of E and is denoted by $d_{\mu_{h\alpha}}(E)$.

Example 4.2: Let $X = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, $\mu_\alpha = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$ and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $M_{\mu_{h\alpha}} = \{a, b, c, d\}$.Then the point $b, c \in M_{\mu_{h\alpha}}$ are μ-$h\alpha$-limit points of a subset $A = \{a, b, d, e\}$ of X. i.e. $d_{\mu_{h\alpha}}(A) = \{b, c\}$. Points $b, c \in M_{\mu_{h\alpha}}$ are μ-$h\alpha$-limit point of a subset $B = \{a, b, c, e\}$ of X.i.e. $d_{\mu_{h\alpha}}(B) = \{b, c\}$. But a,$d \in M_{\mu_{h\alpha}}$ are not μ-$h\alpha$-limit points since $\{a\}$ and $\{d\}$ are μ-$h\alpha$-open sets.

Theorem 4.3: If A, and B are subsets of the GTS X, then the μ-$h\alpha$-derived set has the following properties:

i] $d_{\mu_{h\alpha}}(\emptyset) = \emptyset$,

ii] If $A \subseteq B$ then $d_{\mu_{h\alpha}}(A) \subseteq d_{\mu_{h\alpha}}(B)$,

iii] If $x \in d_{\mu_{h\alpha}}(A)$ then $x \in d_{\mu_{h\alpha}}(A - \{x\})$,

iv] $d_{\mu_{h\alpha}}(A \cap B) \subseteq d_{\mu_{h\alpha}}(A) \cap d_{\mu_{h\alpha}}(B)$,

v] $d_{\mu_{h\alpha}}(A) \cup d_{\mu_{h\alpha}}(B) \subseteq d_{\mu_{h\alpha}}(A \cup B)$.

Remark 4.4: Converse of the theorem 4.3 [iv] and [v] are not true. We justify it through the following example.

Example 4.5: Let $X = \{a, b, c, d, e\}$ **,** $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$,

$\mu_\alpha = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$ and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $M_{\mu_{h\alpha}} = \{a, b, c, d\}$.

For the subset $P = \{a, c, e\} \subseteq X$, $d_{\mu_{h\alpha}}(P) = \{b\}$ and for the subset $Q = \{b, c, d, e\} \subseteq X$, $d_{\mu_{h\alpha}}(Q) = \{\mathrm{b}, c\}$. Thus for the subset $P \cap Q = \{c, e\} \subseteq X$, $d_{\mu_{h\alpha}}(P \cap Q) = \emptyset$. Here we observed that $\{b\} \nsubseteq \emptyset$ i.e. $d_{\mu_{h\alpha}}(P) \cap d_{\mu_{h\alpha}}(Q) \nsubseteq d_{\mu_{h\alpha}}(P \cap Q)$.

For the subset $A = \{a, b\} \subseteq X$, $d_{\mu_{h\alpha}}(A) = \{c\}$ and for the subset $B = \{b, c\} \subseteq X$, $d_{\mu_{h\alpha}}(B) = \{c\}$. Thus for the subset $A \cup B = \{a, b, c\} \subseteq X$, $d_{\mu_{h\alpha}}(A \cup B) = \{b, c\}$. Here we observed that $\{b, c\} \nsubseteq \{c\}$ i.e. $d_{\mu_{h\alpha}}(A \cup B) \nsubseteq d_{\mu_{h\alpha}}(A) \cup d_{\mu_{h\alpha}}(B)$.

Theorem 4.6: If F is μ-$h\alpha$-closed subset of a GTS X then it contain all its μ-$h\alpha$-limit points i.e. $d_{\mu_{h\alpha}}(F) \subseteq F$.

Proof: Suppose F is μ- $h\alpha$-closed set in a GTS X. We have to show that F contain all its μ-$h\alpha$-limit points. Let x be any μ-$h\alpha$-limit point of F. Then to show that $x \in F$. Suppose $x \notin F$.

i.e. $x \in CF$. As F is μ- $h\alpha$-closed set $\Rightarrow$ CF is μ- $h\alpha$-open set. i.e.CF is μ- $h\alpha$-open set containing x. Thus by definition of μ- $h\alpha$-limit point, $CF \cap F$-$\{x\} \neq \emptyset$. This gives contradiction and hence our assumption that $x \notin F$ is wrong. Hence $x \in F$. Thus F contain all its μ-$h\alpha$-limit points.

Remark 4.7: Following example shows that the converse of the above theorem need not be true.

Example 4.8: Let $X = \{a, b, c, d, e\}$ **,** $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, $\mu_\alpha = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$ and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $M_{\mu_{h\alpha}} = \{a, b, c, d\}$.Then the points $b, c \in M_{\mu_{h\alpha}}$ are μ-$h\alpha$-limit points of a subset $A = \{a, b, c, d\}$ of X. i.e. $d_{\mu_{h\alpha}}(A) = \{b, c\}$. Here $d_{\mu_{h\alpha}}(A) \subseteq A$ but A is not μ-$h\alpha$-closed subset of a GTS X.

Definition 4.9: The μ-$h\alpha$-closure of a subset F of GTS X is the intersection of all μ-$h\alpha$-closed set containing F. i.e. $c_{\mu_{h\alpha}}(F) = \cap \{H: H$ is μ-$h\alpha$-closed set, $F \subseteq H\}$.

Theorem 4.10: A subset F of a GTS X is μ-$h\alpha$-closed set iff $F = c_{\mu_{h\alpha}}(F)$.

Proof: Let F be a μ-$h\alpha$-closed subset of a GTS X. By definition, $c_{\mu_{h\alpha}}(F) = \cap \{H: H$- μ-$h\alpha$-closed set, $F \subseteq H\}$ is the smallest μ-$h\alpha$-closed set containing F. i.e. $F \subseteq c_{\mu_{h\alpha}}(F)$ ……………….(7)

But F is μ-$h\alpha$-closed set and $F \subseteq F$ hence $c_{\mu_{h\alpha}}(F) \subseteq F$ ……………….(8) From (7) and (8) we get, $F = c_{\mu_{h\alpha}}(F)$.

Conversely: Let $F = c_{\mu_{h\alpha}}(F)$

As $c_{\mu_{h\alpha}}(F)$ is the smallest μ-$h\alpha$-closed set containing F, hence F is μ-$h\alpha$-closed set.

Theorem 4.11: For any set A in a GTS X, $x \in c_{\mu_{h\alpha}}(A)$ iff $x \in U \in \mu_{h\alpha}$ implies$U \cap A \neq \emptyset$.

Proof: Let $x \in c_{\mu_{h\alpha}}(A)$ and $x \in U \in \mu_{h\alpha}$. Then to show that $U \cap A \neq \emptyset$. Suppose $U \cap A = \emptyset \Rightarrow A \subseteq CU$. As U is μ-$h\alpha$-open set then CU is μ-$h\alpha$-closed set. i.e.CU is μ-$h\alpha$-closed set containing A and $c_{\mu_{h\alpha}}(A)$ is the smallest μ-$h\alpha$-closed set containing A. Hence $c_{\mu_{h\alpha}}(A) \subseteq CU$. $\therefore x \in c_{\mu_{h\alpha}}(A) \subseteq CU \Rightarrow x \in CU$. This gives contradiction since $x \in U$. Hence our assumption that $U \cap A = \emptyset$ is wrong. Thus $U \cap A \neq \emptyset$.

Conversely: Suppose $x \in U \in \mu_{h\alpha}$ implies $U \cap A \neq \emptyset$.

Now $U \cap A \neq \emptyset \Rightarrow U \cap A = \{x\}$ or $U \cap A - \{x\} \neq \emptyset$.

$U \cap A = \{x\} \Rightarrow x \in A \subseteq c_{\mu_{h\alpha}}(A)$ i.e. $x \in c_{\mu_{h\alpha}}(A)$

$U \cap A - \{x\} \neq \emptyset \Rightarrow x \in d_{\mu_{h\alpha}}(A) \subseteq c_{\mu_{h\alpha}}(A)$ i.e. $x \in c_{\mu_{h\alpha}}(A)$.

Thus in both cases we get $x \in c_{\mu_{h\alpha}}(A)$.

Remark 4.12: μ-$h\alpha$-closure operators on a GTS X obey the following properties, for any subset E of X.

i] $c_{\mu_{h\alpha}}(\emptyset) = X - M_{\mu_{h\alpha}}$ except for $X = M_{\mu_{h\alpha}}$,

ii] $E \subseteq c_{\mu_{h\alpha}}(E)$,

iii] $c_{\mu_{h\alpha}}\left(c_{\mu_{h\alpha}}(E)\right) = c_{\mu_{h\alpha}}(E)$,

iv] $c_{\mu_{h\alpha}}(E)$ is the smallest μ-$h\alpha$-closed set containing E.

v] $E \cup d_{\mu_{h\alpha}}(E) \subseteq c_{\mu_{h\alpha}}(E)$.

Remark 4.13: Converse of the Remark 4.12[v] is not true. We justify it through the following example.

Example 4.14: Let $X = \{a, b, c, d, e\}$ **,** $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$,

$\mu_{\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$ and

$\mu_{h\alpha} = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$, $M_{\mu_{h\alpha}} = \{a, b, c, d\}$.
$\mathcal{F}_{\mu_{h\alpha}} = \{X, \{b, c, d, e\}, \{a, b, c, e\}, \{b, c, e\}, \{d, e\}, \{c, e\}, \{a, e\}, \{e\}\}$ For the subset $A = \{a, b\} \subseteq X$, $d_{\mu_{h\alpha}}(A) = \{c\}$ and $c_{\mu_{h\alpha}}(A) = X \cap \{a, b, c, e\} = \{a, b, c, e\}$. Here we observed that $\{a, b\} \cup \{c\} = \{a, b, c\} \nsubseteq \{a, b, c, e\}$ i.e. $A \cup d_{\mu_{h\alpha}}(A) \subsetneq c_{\mu_{h\alpha}}(A)$.

Theorem 4.15: Relation between μ-$h\alpha$-**i**nterior and μ-$h\alpha$- closure operator in a GTS X.

Let E be any subset of a GTS X. Then

i] $i_{\mu_{h\alpha}}(E) = Cc_{\mu_{h\alpha}}(CE)$. i.e. $i_{\mu_{h\alpha}}(E) = X - c_{\mu_{h\alpha}}(X - E)$,

ii] $c_{\mu_{h\alpha}}(E) = Ci_{\mu_{h\alpha}}(CE)$ i.e.$c_{\mu_{h\alpha}}(E) = X - i_{\mu_{h\alpha}}(X - E)$.

Proof: We know in a GTS(X,μ), $i_{\mu_{h\alpha}}(E) \subseteq E$.

$\Rightarrow C(i_{\mu_{h\alpha}}(E)) \supseteq C(E)$ (by taking the complement on both sides)

$\Rightarrow c_{\mu_{h\alpha}}(C\left(i_{\mu_{h\alpha}}(E)\right)) \supseteq c_{\mu_{h\alpha}}(C(E))$ (by taking the μ-$h\alpha$-closures on both sides)

As $i_{\mu_{h\alpha}}(E)$ is μ-$h\alpha$-open set thus $C\left(i_{\mu_{h\alpha}}(E)\right)$ is μ-$h\alpha$-closed set and hence

$c_{\mu_{h\alpha}}\left(C\left(i_{\mu_{h\alpha}}(E)\right)\right) = C\left(i_{\mu_{h\alpha}}(E)\right)$

$\Rightarrow C\left(i_{\mu_{h\alpha}}(E)\right) \supseteq c_{\mu_{h\alpha}}(C(E))$

$\Rightarrow i_{\mu_{h\alpha}}(E) \subseteq C(c_{\mu_{h\alpha}}(C(E)))$ (by taking the complement on both sides) …………..(9)

As $c_{\mu_{h\alpha}}(CE)$ is the smallest μ-$h\alpha$-closed set containing CE i.e. $CE \subseteq c_{\mu_{h\alpha}}(CE)$

$\Rightarrow C(c_{\mu_{h\alpha}}(C(E)))$ is μ-$h\alpha$-open set and $C\left(c_{\mu_{h\alpha}}(C(E))\right) \subseteq C(CE) = E$

i.e. $C(c_{\mu_{h\alpha}}(C(E)))$ is μ-$h\alpha$-open set contained in E and $i_{\mu_{h\alpha}}(E)$ is the largest μ-$h\alpha$-open set contained in E.Hence $C(c_{\mu_{h\alpha}}(C(E))) \subseteq i_{\mu_{h\alpha}}(E)$ …………..(10)

Thus from (9) and (10) we get, $i_{\mu_{h\alpha}}(E) = Cc_{\mu_{h\alpha}}(CE) = X - c_{\mu_{h\alpha}}(X-E)$ …………..(11)

In (11) replace E by CE and taking complement on both sides we get,

$C(i_{\mu_{h\alpha}}(CE)) = c_{\mu_{h\alpha}}(E)$ i.e. $c_{\mu_{h\alpha}}(E) = X - (i_{\mu_{h\alpha}}(X-E)$.

Theorem 4.16: If A and B be any two subsets of a GTS X then

i] If $A \subseteq B$ then $c_{\mu_{h\alpha}}(A) \subseteq c_{\mu_{h\alpha}}(B)$,

ii] $c_{\mu_{h\alpha}}(A) \cup c_{\mu_{h\alpha}}(B) \subseteq c_{\mu_{h\alpha}}(A \cup B)$,

iii] $c_{\mu_{h\alpha}}(A \cap B) \subseteq c_{\mu_{h\alpha}}(A) \cap c_{\mu_{h\alpha}}(B)$.

Proof: [i] Let A and B be any two subsets of a GTS Xsuch that $A \subseteq B$.

By theorem 4.15, $c_{\mu_{h\alpha}}(A) = X - i_{\mu_{h\alpha}}(X-A)$.

As $A \subseteq B \Rightarrow X-B \subseteq X-A$ $\Rightarrow i_{\mu_{h\alpha}}(X-B) \subseteq i_{\mu_{h\alpha}}(X-A)$ $\Rightarrow X - i_{\mu_{h\alpha}}(X-A) \subseteq X - i_{\mu_{h\alpha}}(X-B)$.i.e. $c_{\mu_{h\alpha}}(A) \subseteq c_{\mu_{h\alpha}}(B)$.

The proof of [ii] and [iii] follows clearly from [i].

Remark 4.17: Converse of the theorem 4.16 [ii] and [iii] are not true. We justify it through the following example.

Example 4.18: Let $X = \{a,b,c,d,e\}$, $\mu = \{\emptyset,\{a\},\{d\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\}\}$,

$\mu_\alpha = \{\emptyset,\{a\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{a,b,c,d\}\}$ and

$\mu_{h\alpha} = \{\emptyset,\{a\},\{d\},\{a,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}$, $M_{\mu_{h\alpha}} = \{a,b,c,d\}$.

For the subset $A = \{a\} \subseteq X$, $c_{\mu_{h\alpha}}(A) = \{a,e\}$ and for the subset $B = \{d\} \subseteq X$, $c_{\mu_{h\alpha}}(B) = \{\mathrm{d},\mathrm{e}\}$. Thus for the subset $A \cup B = \{a,d\} \subseteq X$, $\mathrm{c}_{\mu_{h\alpha}}(A \cup B) = \mathrm{X}$. Here we observed that$\{a,e\} \cup \{\mathrm{d},\mathrm{e}\} = \{\mathrm{a},\mathrm{d},\mathrm{e}\} \subsetneq X$. i.e. $c_{\mu_{h\alpha}}(A) \cup c_{\mu_{h\alpha}}(B) \subsetneq c_{\mu_{h\alpha}}(A \cup B)$.

Also for the subset $P = \{b,c,d\} \subseteq X$, $c_{\mu_{h\alpha}}(P) = \{b,c,d,e\}$ and for the subset $Q = \{\mathrm{a},d,e\} \subseteq X$, $c_{\mu_{h\alpha}}(Q) = \mathrm{X}$. Thus for the subset $P \cap Q = \{d\} \subseteq X$, $\mathrm{c}_{\mu_{h\alpha}}(P \cap Q) = \{\mathrm{d},\mathrm{e}\}$. Here we observed that $\{\mathrm{d},\mathrm{e}\} \subsetneq \{b,c,d,e\} \cap X = \{\mathrm{b},\mathrm{c},\mathrm{d},\mathrm{e}\}$. i.e. $c_{\mu_{h\alpha}}(P \cap Q) \subsetneq c_{\mu_{h\alpha}}(P) \cap c_{\mu_{h\alpha}}(Q)$.

Theorem 4.19: If $x \notin F$,where F is a μ-$h\alpha$-closed subset of a GTS X, then there exists μ-$h\alpha$-open set G such that $x \in G \subseteq CF$.

Proof: Let $x \notin F$, μ-$h\alpha$-closed subset of a GTS X. As F is μ-$h\alpha$-closed set thus by theorem4.10, $F = c_{\mu_{h\alpha}}(F)$. By theorem 4.15, $c_{\mu_{h\alpha}}(F) = X - i_{\mu_{h\alpha}}(X - F)$.

Thus $x \notin X - i_{\mu_{h\alpha}}(X - F) \Rightarrow x \in i_{\mu_{h\alpha}}(X - F)$

i.e. there exists μ-$h\alpha$-open set $i_{\mu_{h\alpha}}(X - F)$ such that $x \in i_{\mu_{h\alpha}}(X - F) \subseteq X - F = CF$.

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