On Point Wise Products of Uniformly Continuous Functions on Uniform Space

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Abstract: *In this paper we obtain the sufficient conditions on a uniform space $(X,\mathcal{U})$ for which $\mathcal{UC}(X)$, the family of all uniformly continuous functions on X is an algebra. It is proved that Theorem A: If $(X,\mathcal{U})$ is a uniformly continuous uniform space then $\mathcal{UC}(X)$ is an algebra.*

Theorem B:If $(X,\mathcal{U})$ is precompact uniform space. Then $\mathcal{UC}(X)$ is an algebra.

We prove that $\mathcal{UC}_{\mathbb{R}}(X)$, the family of all uniformly continuous real valued functions on X is a lattice of functions which need not be a complete lattice in the sense that every subset of $\mathcal{UC}_{\mathbb{R}}(X)$ may not have supremum or infimum in $\mathcal{UC}_{\mathbb{R}}(X)$ by providing a counter example.

Keywords: *Uniform space, Uniformly continuous space, Lattice, Complete Lattice.*

Introduction: Let $(X,\mathcal{U})$ be a uniform space. We shall denote by $\mathcal{UC}(X)$, the family of all uniformly continuous functions on X and $\mathcal{UC}_{\mathbb{R}}(X)$, the family of all uniformly continuous real valued functions on X.

Definition:1] Lattice: A lattice is a partially ordered set in which every pair of elements has both an infimum and a supremum.

Definition:2] Complete Lattice: A complete lattice is a partially ordered set in which all susets have both a supremum and an infimum.

Theorem 3: Let $(X,\mathcal{U})$ be a uniform space. Let $\mathcal{UC}(X)$ be the family of all uniformly continuous functions on X. Then for any $f,g \in \mathcal{UC}(X)$

1] $f \pm g$ is uniformly continuous.

2] αf is uniformly continuos, where α is any scalar

3] $|f|$ is uniformly continuous.

4]] If f, $g \in \mathcal{UC}_{\mathbb{R}}(X)$ then $\max(f,g)$, $\min(f,g)$ are uniformly continuous.

Hence $\mathcal{UC}(X)$ is a complex vector space.

Proof:

1] We show that the mapping $f+g$ is uniformly continuous.

Let $\varepsilon > 0$ be given. Then $\exists\ U \in \mathcal{U}$ such that whenever $(x,y) \in U \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2}$.

Also $\exists\ V \in \mathcal{U}$ such that whenever $(x,y) \in V \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2}$.

Now $U,V \in \mathcal{U} \Rightarrow W = U \cap V \in \mathcal{U}$.

Then for any $(x,y) \in W \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2}$ and $|g(x)-g(y)| < \frac{\varepsilon}{2}$.

Thus $|(f+g)(x)-(f+g)(y)| = |f(x)+g(x)-f(y)-g(y)|$

$$= |f(x)-f(y)+g(x)-g(y)|$$
$$\leq |f(x)-f(y)| + |g(x)-g(y)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow f+g$ is uniformly continuous.

Similarly, $f-g$ is uniformly continuous.

2] We show that αf is uniformly continuous, where α is any scalar.

Case I: $\alpha = 0$.

Then $\alpha f = 0$ is uniformly continuous.

Case II: $\alpha \neq 0$.

Let $\varepsilon > 0$ then $\exists\, U \in \mathcal{U}$ such that $(x, y) \in U \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{|\alpha|}$

Then $|\alpha f(x) - \alpha f(y)| = |\alpha(f(x) - f(y))|$

$=|\alpha||f(x) - f(y)|$

$< |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon.$

$\Rightarrow \alpha f$ is uniformly continuos.

3] We show that $|f|$ is uniformly continuous.

Let $\varepsilon > 0$ then $\exists\, U \in \mathcal{U}$ such that $(x, y) \in U \Rightarrow |f(x) - f(y)| < \varepsilon$.

We know, $\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)| < \varepsilon$.

i.e. for a given $\varepsilon > 0$, $\exists$ $U \in \mathcal{U}$ such that $(x, y) \in U \Rightarrow \big||f(x)| - |f(y)|\big| < \varepsilon$.

Thus $|f|$ is uniformly continuous.

4] Put $h = \max(f, g)$.

Firstly we show that for any two real numbers α, β, $\max(\alpha, \beta) = \frac{\alpha+\beta+|\alpha-\beta|}{2}$.

Suppose $\alpha \geq \beta$ then $\max(\alpha, \beta) = \alpha$ ……..…(i)

and $\frac{\alpha+\beta+|\alpha-\beta|}{2} = \frac{\alpha+\beta+\alpha-\beta}{2} = \frac{2\alpha}{2} = \alpha$ ………....(ii)

Similarly if $\beta \geq \alpha$ then $\max(\alpha, \beta) = \beta$ …..……(iii)

and $\frac{\alpha+\beta+|\alpha-\beta|}{2} = \frac{\alpha+\beta-\alpha+\beta}{2} = \frac{2\beta}{2} = \beta$ ………..(iv)

From (i),(ii),(iii) and (iv) we get, $\max(\alpha, \beta) = \frac{\alpha+\beta+|\alpha-\beta|}{2}$ $\forall\, \alpha, \beta$.

If f, g are uniformly continuous real valued functions on X then

$h(x) = \max(f(x), g(x)) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$, $x \in X$, is also uniformly continuous real valued function on X, by [1] and [3].

Similarly, $k = \min(f, g) = -\max(-f, -g)$ is also uniformly continuous real valued functions on X.

Now we give sufficient conditions on $(X, \mathcal{U})$ such that $\mathcal{UC}(X)$ is an algebra.

Theorem A: If $(X, \mathcal{U})$ is a uniformly continuous uniform space then $\mathcal{UC}(X)$ is an algebra.

Proof: By theorem 1, $\mathcal{UC}(X)$ is closed under addition and scalar multiplication.

Now we show that it is closed under multiplication.

Let $f, g \in \mathcal{UC}(X)$. i.e. f, g are uniformly continuous functions on X. Hence they are continuous on X. Since product of continuous functions is continuous, fg is continuous function on X.

As X is uniformly continuous uniform space, fg is uniformly continuous function on X.Hence $fg \in \mathcal{UC}(X)$. Thus $\mathcal{UC}(X)$ is an algebra.

Lemma 4: For a uniform space every uniformly continuous function maps precompact set onto precompact set.

Proof: Let $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniformly continuous function and E be a precompact subset of X. We show that $f(E)$ is a precompact subset of Y.

Let $V \in \mathcal{V}$ be given. As f is a uniformly continuous function on X, there exists $U \in \mathcal{U}$ such that $(x, y) \in U \Rightarrow (f(x), f(y)) \in V$ ……….(1)

E is precomact thus for $U \in \mathcal{U}$ there exists a finite subset $\{x_1, x_2, \dots \dots, x_n\}$ in X such that

$E \subset \bigcup_{i=1}^{n} U[x_i]$ ……….(2)

Now we show that $f(E) \subset \bigcup_{i=1}^{n} V[f(x_i)]$.

Let $y \in f(E)$. Then $y = f(x)$ for some $x \in E$.

Now $x \in E \Rightarrow (x, x_i) \in U$ for some i, $1 \leq i \leq n$ from (2)

$\Rightarrow (f(x), f(x_i)) \in V$ from (1) i.e. $f(x) \in V[f(x_i)]$ for the above i.

i.e. $y \in V[f(x_i)] \subset \cup_{i=1}^{n} V[f(x_i)]$

Thus $f(E) \subset \cup_{i=1}^{n} V[f(x_i)]$.

i.e. for $V \in \mathcal{V}$ there exists a finite subset $\{f(x_1), f(x_2), \dots \dots, f(x_n)\}$ in Y such that

$f(E) \subset \cup_{i=1}^{n} V[f(x_i)]$.

Thus $f(E)$ is precompact.

Theorem B: If $(X, \mathcal{U})$ is precompact uniform space. Then $\mathcal{UC}(X)$ is an algebra.

Proof: By theorem 3, we see that $\mathcal{UC}(X)$ is closed under addition and scalar multiplication.

Now we show that it is closed under point wise product.

Let $f, g \in \mathcal{UC}(X)$. We show that $fg \in \mathcal{UC}(X)$.

As X is precompact uniform space. By above lemma 4, $f(X)$ and $g(X)$ are precompact and hence are bounded. Thus there exists $k_1, k_2 > 0$ such that $|f(x)| \le k_1$ and $|g(x)| \le k_2$.

Now we show that $fg \in \mathcal{UC}(X)$.

Let $r > 0$ be given. Then there exists $U, V \in \mathcal{U}$ such that

$(x, y) \in U \quad \Rightarrow \quad |f(x) - f(y)| < \frac{r}{2k_2}$ and $(x, y) \in V \quad \Rightarrow \quad |g(x) - g(y)| < \frac{r}{2k_1}$.

Now $U, V \in \mathcal{U} \Rightarrow \quad U \cap V \in \mathcal{U}$.

Then for $(x, y) \in U \cap V \Rightarrow (x, y) \in U$ and $(x, y) \in V$

$\Rightarrow \quad |f(x) - f(y)| < \frac{r}{2k_2}$ and $|g(x) - g(y)| < \frac{r}{2k_1}$

Consider,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &= \big|f(x)\big(g(x) - g(y)\big) + (f(x) - f(y))g(y)\big| \\ &\le |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)| \\ &\le k_1 \frac{r}{2k_1} + \frac{r}{2k_2} k_2 = r. \end{aligned}$$

i.e. for $r > 0$, there exists $U \cap V \in \mathcal{U}$ such that

$(x, y) \in U \cap V \Rightarrow |f(x)g(x) - f(y)g(y)| < r$.

$\Rightarrow fg \in \mathcal{UC}(X)$.

- $\mathcal{UC}_{\mathbb{R}}(X)$ is the family of all uniformly continuous real valued functions on X. It is a partially ordered set by defining a relation $f \ge g \quad \Leftrightarrow \quad f(x) \ge g(x), \ \forall\, x \in X$.
  Thus by above theorem 3[4], $\mathcal{UC}_{\mathbb{R}}(X)$ is a lattice of functions. But $\mathcal{UC}_{\mathbb{R}}(X)$ need not be a complete lattice in the sense that every subset of $\mathcal{UC}_{\mathbb{R}}(X)$ may not have supremum or infimum in $\mathcal{UC}_{\mathbb{R}}(X)$.
  We prove by giving a counter example.

Ex 5. Let $X = [0,1]$ and $\mathcal{U}$ be the uniformity on X defined by the pseudo metric d, with

$d(x, y) = \int_0^1 |x(t) - y(t)| dt$. Then $\mathcal{UC}_{\mathbb{R}}(X)$ is a lattice which is not a complete lattice.

Proof: For each $m \in \mathbb{N}$, x_m is a function defined on X as

$x_m(t) = 0 \quad$ if $t \in \left[0, \frac{1}{2}\right]$,

$x_m(t) = 1 \quad$ if $t \in [a_m, 1]$

where $a_m = \frac{1}{2} + \frac{1}{m}$.

$x_m(t)$ is linear joining the points $(\frac{1}{2}, 0)$ and $(a_m, 1)$ for $t \in [\frac{1}{2}, a_m]$

Then the function $x_m \in \mathcal{UC}_{\mathbb{R}}(X)$.

Let $A = \{x_m \in \mathcal{UC}_{\mathbb{R}}(X) \ : \ m \ge 1\}$. Take $x = \sup_n \{x_n : n \ge 1\}$.

Thus $x(t) = \sup_n \{x_n(t) : n \ge 1\}$, $t \in [0,1]$.

Then for $t \in [0, \frac{1}{2}]$, $x_n(t) = 0$ for all $n \geq 1$.

$\therefore x(t) = \sup_n \{x_n(t) : n \geq 1\} = 0$ if $t \in [0, \frac{1}{2}]$, ...(1)

For $t \in (\frac{1}{2}, 1]$, $\exists\ m \in \mathbb{N}$ such that $\frac{1}{2} < \frac{1}{2} + \frac{1}{m} < t \leq 1$

Then $\forall\ n \geq m$, $\frac{1}{n} < \frac{1}{m}$.

$\therefore\ \frac{1}{n} + \frac{1}{2} < \frac{1}{m} + \frac{1}{2}$ ....(2)

Thus $\forall\ n \geq m$, $\frac{1}{2} < \frac{1}{n} + \frac{1}{2} < \frac{1}{2} + \frac{1}{m} < t \leq 1$ ....(3)

i.e. For $t \in (\frac{1}{2}, 1]$, $\exists\ m \in \mathbb{N}$ such that $\forall\ n \geq m$, $t \in (a_n, 1]$.

$\therefore\ x_n(t) = 1$

$\therefore\ x(t) = \sup_n \{x_n(t) : n \geq 1\} = 1$ if $t \in \left(\frac{1}{2}, 1\right]$ ....(4)

From (3) and (4) we get x is not a continuous function. Hence $x \notin \mathcal{UC}_{\mathbb{R}}(X)$.

i.e. $\mathcal{UC}_{\mathbb{R}}(X)$ does not contain supremum of A. Thus $\mathcal{UC}_{\mathbb{R}}(X)$ is not a complete lattice.

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