On Equinormal Proximity Space and Uniformly Continuous Uniform Space

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Abstract: In this paper we obtain the characterization of uniformly continuous pseudo metric spaces in terms of the associated equinormal proximity spaces. The precise result is the following.

If (X, d) is a pseudo metric space and $\delta = \delta(d)$ is the associated proximity on X, then (X, d) is uniformly continuous if and only if (X, δ) is equinormal proximity space.

We also characterize equinormality of proximity space associated with normal uniform space in terms of proximity of continuous mapping. Precisely the following is proved.

If $(X, \mathcal{U})$ is a normal uniform space and δ is the associated proximity on X then (X, δ) is equinormal proximity space iff every continuous real valued function on X is a proximity mapping. Here the proximity δ_1 on $\mathbb{R}$ is defined as $A\delta_1 B \Leftrightarrow d(A, B) = \inf\{|x - y| \, : x \in A, y \in B\} = 0$.

Also we obtain the sufficient conditions for a uniform space to define equinormal proximity. The precise results are as follows.

Let $(X, \mathcal{U})$ be a uniform space and δ be the associated proximity on X. If for any two non empty disjoint closed sets at least one is compact, then (X, δ) is equinormal.

For a normal uniform space $(X, \mathcal{U})$ and the associated proximity δ, if $(X, \mathcal{U})$ is uniformly continuous space then (X, δ) is equinormal.

Key words: Uniformly continuous space, Proximity space, Equinormal Proximity space and Proximity mapping.

1. Characterization of uniformly continuous pseudo metric spaces in terms in terms of proximity :

Definition 1.1:

Equinormal proximity space: A proximity space (X, δ) is equinormal iff $A\delta B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$.

Theorem 1.2:

Suppose (X, d) is a pseudo metric space. Then (X, d) is uniformly continuous space if and only if $\bar{A} \cap \bar{B} = \emptyset \Leftrightarrow d(A, B) > 0$.

This is the theorem4, p. 1801[5].

Proposition 1.3:

Let (X, d) be a pseudo metric space. Let δ_1 be a binary relation defined on the power set of X by $A\delta_1 B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$ and δ_2 be a binary relation defined on the power set of X by

$A\delta_2 B \Leftrightarrow d(A, B) = 0$. Then

1] δ_1 is a proximity on X.

2] δ_2 is a proximity on X.

3] For any $A, B \subset X$, if $A\ \delta_1\ B$ then $A\ \delta_2\ B$ but not conversely.

4] $\mathcal{T} = \mathcal{T}(\delta_1) = \mathcal{T}(\delta_2)$

Where, $\mathcal{T}$- the topology induced by the pseudo metric d

$\mathcal{T}(\delta_1)$- the topology induced by the proximity δ_1

$\mathcal{T}(\delta_2)$-the topology induced by the proximityδ_2

This is the theorem2.11 and remark 2.18 [4].

Proposition 1.4:

If (X, d) is a pseudo metric space and δ_1, δ_2 are proximities defined on X as

$A\delta_1 B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$ and $A\delta_2 B \Leftrightarrow d(A, B) = 0$.

Then (X, d) is uniformly continuous space if and only if $\delta_1 = \delta_2$.

Proof: By theorem 1.2,

(X, d) is uniformly continuous space $\Leftrightarrow \bar{A} \cap \bar{B} = \emptyset \Rightarrow d(A, B) > 0$

$\Leftrightarrow d(A, B) = 0 \Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$ (Contrapositively)

$\Leftrightarrow A\delta_2 B \Rightarrow A\delta_1 B$ ……….(1)

By proposition 1.3, $\delta_1 > \delta_2$ i.e. $A\delta_1 B \Rightarrow A\delta_2 B$ ……….(2)

Thus from (1) and (2) we get,

(X, d) is uniformly continuous space $\Leftrightarrow A\delta_1 B \Leftrightarrow A\delta_2 B \Leftrightarrow \delta_1 = \delta_2$.

Theorem 1.5:

If (X,d) is a pseudo metric space and δ_2 is the associated proximity on X. Then (X,d) is uniformly continuous if and only if (X,δ_2) is equinormal.

Proof: By above proposition 1.4,

(X,d) is uniformly continuous $\Leftrightarrow A\delta_2 B \Leftrightarrow A\delta_1 B$

$\Leftrightarrow A\delta_2 B \Leftrightarrow \bar{A} \cap \bar{B} \neq \emptyset$ (by definition of the proximity δ_1)

$\Leftrightarrow (X,\delta_2)$ is equinormal (by definition of equinormal space).

2. Characterization of Equinormal proximity spaces:

Definition 2.1:

Proximity Mapping: Let (X,δ_1) and (Y,δ_2) be two proximity spaces. A function $f: X \to Y$ is said to be a proximity mapping if and only if $A\,\delta_1 B \Rightarrow f(A)\delta_2\, f(B)$.

Lemma 2.2:

For subsets A and B of a proximity space(X,δ), $A\delta B \Leftrightarrow \bar{A}\delta\bar{B}$, where the closure is taken with respect to $\mathcal{T}(\delta)$. This is the lemma 2.8,p.12[4].

Theorem 2.3 :

Every uniform space $(X,\mathcal{U})$ has an associated proximity $\delta = \delta(\mathcal{U})$ defined by

$A\delta B \Leftrightarrow (A \times B) \cap U \neq \emptyset,$ for every $U \in \mathcal{U}.$

This is the theorem 10.2, p. 64[4].

Theorem 2.4:

Let $(X,\mathcal{U})$ be a normal uniform space and $\delta = \delta(\mathcal{U})$. If (X,δ) is equinormal proximity space then every continuous real valued function on X is a proximity mapping, where the proximity δ_1 on $\mathbb{R}$ is any proximity compatible with usual topology on $\mathbb{R}$.

Proof: Let $f:(X,\mathcal{U})\to(\mathbb{R},\mathcal{V})$ be a continuous real valued function. We show that f is a proximity mapping.

let $A,B\subset X$ such that $A\delta B$.

$\Rightarrow\ \bar{A}\cap\bar{B}\neq\emptyset$ (since (X,δ) is equinormal)

$\Rightarrow\ f(\bar{A})\cap f(\bar{B})\neq\emptyset$

$\Rightarrow\ \overline{f(A)}\cap\overline{f(B)}\neq\emptyset$ (since f is continuous $f(\bar{A})\subset\overline{f(A)}$ and $f(\bar{B})\subset\overline{f(B)}$)

$\Rightarrow \overline{f(A)}\delta_1\overline{f(B)}$ (by proximity axiom)

$\Rightarrow f(A)\delta_1 f(B)$ (by Lemma 2.2) i.e. $A\delta B\Rightarrow f(A)\delta_1 f(B)$.`

Theorem 2.5:

Let $(X,\mathcal{U})$ be a normal uniform space and $\delta=\delta(\mathcal{U})$. If every continuous real valued function on X is a proximity mapping then (X,δ) is equinormal.

Here the proximity δ_1 on $\mathbb{R}$ is defined as $A\delta_1 B\Leftrightarrow \inf\{|x-y| : x\in A, y\in B\}=0,\ \ A,B\subset\mathbb{R}$.

Proof: To show that (X,δ) is equinormal, we show that $\bar{A}\cap\bar{B}=\emptyset\Rightarrow A\not\delta B$.

Let $A,B\subset X$ such that $\bar{A}\cap\bar{B}=\emptyset$. Since X is normal, there exists a continuous function $f:X\to\mathbb{R}$ such that $f(\bar{A})=0$ and $f(\bar{B})=1$.

Suppose $A\delta B$. Then by Lemma 2.2, $\bar{A}\delta\bar{B}$. As f is continuous, by hypothesis f is a proximity mapping. Thus $\bar{A}\delta\bar{B}\Rightarrow\ f(\bar{A})\delta_1 f(\bar{B})$

$\Rightarrow \left(f(\bar{A})\times f(\bar{B})\right)\cap V_{d,r}\neq\emptyset,\quad \forall r>0$. Here $V_{d,r}=\{(x,y) : |x-y|<r\}$

Thus for each $n\in\mathbb{N}$, $(f(\bar{A})\times f(\bar{B}))\cap V_{d,\frac{1}{n}}\neq\emptyset$.

$\therefore$ for $n=2$, there exists $x\in\bar{A}$ and $y\in\bar{B}$ such that $|f(x)-f(y)|<\frac{1}{2}$.

But $x\in\bar{A}$ and $y\in\bar{B}\Rightarrow f(x)=0$ and $f(y)=1$ then $|f(x)-f(y)|=|0-1|=1\nless\frac{1}{2}$.

This contradiction proves that $A\not\delta B$.

Combining theorem 2.4 & theorem 2.5 we get the following result.

Theorem 2.6:

Let $(X, \mathcal{U})$ be a normal uniform space and $\delta = \delta(\mathcal{U})$. Then (X, δ) is equinormal proximity space iff every continuous real valued function on X is a proximity mapping. Here the proximity δ_1 on $\mathbb{R}$ is defined as $A\delta_1 B \Leftrightarrow d(A, B) = \inf\{|x - y| \, : x \in A, y \in B\} = 0$.

3. Sufficient conditions for a Uniform space to define Equinormal Proximity Space:

Theorem 3.1:

Let $(X, \mathcal{U})$ be a uniform space. Let $\delta = \delta(\mathcal{U})$. If for any two non empty disjoint closed subsets A, B of X at least one is compact then $A\delta B \Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$. ie. (X, δ) is equinormal.

Proof: We show that $A\delta B \Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$.

Suppose $\bar{A} \cap \bar{B} = \emptyset$ but $A\delta B$. By assumption we may assume that $\bar{A}$ is compact.

Since $A\delta B$,$(A \times B) \cap U \neq \emptyset$, $\forall U \in \mathcal{U}$. Thus for each $U \in \mathcal{U}$ we may choose a point $(x_U, y_U) \in U$ such that $(x_U, y_U) \in (A \times B) \cap U$.

Thus we get the net $\{x_U : U \in \mathcal{U}, \geq\}$ and $\{y_U : U \in \mathcal{U}, \geq\}$ in A and B respectively such that $(x_U, y_U) \in U$. The net $\{x_U : U \in \mathcal{U}, \geq\}$ is in A and $\bar{A}$ is compact.

Thus there is a subnet $\{z_P : P \in E, \geq\}$ of $\{x_U : U \in \mathcal{U}, \geq\}$ which converges to z in $\bar{A}$.

i.e. for each $U \in \mathcal{U}$ there is $P_1 \in E$ such that if $Q \in E$ and $Q \geq P_1$ then $(z_Q, z) \in U$. ……(3)

As $\{z_P : P \in E, \geq\}$ is a subnet of the net $\{x_U : U \in \mathcal{U}, \geq\}$, there is a function $N: E \to \mathcal{U}$ such that $x \circ N = z$ i.e. $x_{N_P} = z_P$ for all $P \in E$.

Also for each $U \in \mathcal{U}$ there is $P_2 \in E$ with the property that if $Q \geq P_2$ then $N_Q \geq U$. …….(4)

Now we show that $\{(y \circ N)(Q) : Q \in E, \geq\}$ converges to z.

Let $U \in \mathcal{U}$.

Then $\exists\, V \in \mathcal{U}$ such that $V \circ V \subset U$.

Then from (3) for $V \in \mathcal{U}, \exists\, P_1 \in E$ such that if $Q \in E$ and $Q \geq P_1$ then $(z_Q, z) \in V$.

Also from (4) for $V \in \mathcal{U}, \exists\, P_2 \in E$ such that if $Q \in E$ and $Q \geq P_2$ then $N_Q \geq V$.

Now for $P_1, P_2 \in E, \exists\, P \in E$ such that $P \geq P_1$ and $P \geq P_2$ (by definition of directed set).

Then for $Q \geq P$ we have $N_Q \geq V$ and $(z_Q, z) \in V$. ………….(5)

$Q \geq P \Rightarrow N_Q \geq V \Rightarrow N_Q[p] \subset V[p]$ for all $p \in X$ ………….(6)

Now $z_Q = x \circ N_Q = x_{N_Q} \in A$

Thus for $x_{N_Q} \in A$ there is $y_{N_Q} \in B$ such that $(x_{N_Q}, y_{N_Q}) \in N_Q$

$\Rightarrow y_{N_Q} \in N_Q[x_{N_Q}] \subset V\left[x_{N_Q}\right]$ from (6)

$\Rightarrow y_{N_Q} \in V\left[x_{N_Q}\right] \Rightarrow (x_{N_Q}, y_{N_Q}) \in V$

i.e. $(z_Q, y_{N_Q}) \in V$ and V is symmetric thus $(y_{N_Q}, z_Q) \in V$ ....……..(7)

Thus from (5) and (7) we get $\left(y_{N_Q}, z\right) = (y_{N_Q}, z_Q,) \circ (z_Q, z) \in V \circ V \subset U \Rightarrow \left(y_{N_Q}, z\right) \in U$

i.e. for each $U \in \mathcal{U}$ there is $P \in E$ such that if $Q \in E$ and $Q \geq P$ then $\left(y_{N_Q}, z\right) \in U$.

Thus the net $\{y_{N_Q} : Q \in E, \geq\}$ in B converges to z.

Hence $z \in \bar{B}$. i.e. $z \in \bar{A} \cap \bar{B} \Rightarrow \bar{A} \cap \bar{B} \neq \emptyset$.

This gives contradiction to the given condition. Hence our assumption that $A\delta B$ is wrong.

Thus $A \not\delta B$.

Theorem 3.2:

Suppose $(X, \mathcal{U})$ is a normal uniform space and $\delta = \delta(\mathcal{U})$ is an associated proximity on X. If $(X, \mathcal{U})$ is uniformly continuous space then (X, δ) is equinormal.

Proof: Let $A, B \subset X$ such that $\bar{A} \cap \bar{B} = \emptyset$. Then we show that $A \not\delta B$.

Suppose $A\delta B$ and $\delta = \delta(\mathcal{U})$. Then $(A \times B) \cap U \neq \emptyset, \ \forall\, U \in \mathcal{U}$.

So we may choose a point $(x_U, y_U) \in (A \times B) \cap U, \quad \forall\, U \in \mathcal{U}$.

Thus we get a net $\{x_U : U \in \mathcal{U}, \geq\}$ in A and $\{y_U : U \in \mathcal{U}, \geq\}$ in B such that

$(x_U, y_U) \in (A \times B) \cap U, \quad \forall\, U \in \mathcal{U}$ ……..(8)

Also $\bar{A} \cap \bar{B} = \emptyset$ and X is normal, there exist a continuous function $f: X \to \mathbb{R}$ such that

$f(\bar{A}) = 0$ and $f(\bar{B}) = 1$.

As X is uniformly continuous, the continuous function f is uniformly continuous.

i.e. for every $r > 0$, there exists $U \in \mathcal{U}$ such that

whenever $(x, y) \in U \Rightarrow |f(x) - f(y)| < r$ …....(9)

Thus for $r = \frac{1}{2} > 0$, there exists $U_0 \in \mathcal{U}$ such that $(x, y) \in U_0 \Rightarrow |f(x) - f(y)| < \frac{1}{2}$.

But for $U_0 \in \mathcal{U}$ there exists $x_{U_0} \in A$ and $y_{U_0} \in B$ such that $(x_{U_0}, y_{U_0}) \in (A \times B) \cap U_0$ from(8)

Hence from (9), $\left|f\left(x_{U_0}\right) - f(y_{U_0})\right| < \frac{1}{2}$.

But $f\left(x_{U_0}\right) = 0$ and $f\left(y_{U_0}\right) = 1$, then $\left|f\left(x_{U_0}\right) - f(y_{U_0})\right| = |0 - 1| \nless \frac{1}{2}$.

This gives contradiction. Hence our assumption that $A\delta B$ is wrong. Thus $A\not\delta B$.

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