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CHARACTERIZATION OF UNIFORMLY CONTINUOUS PSEUDO METRIC SPACES

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ABSTRACT

A pseudo metric space (X, d) is called a uniformly continuous if every continuous real valued function on (X, d) is uniformly continuous. In this paper we obtain the characterizations of uniformly continuous pseudo metric spaces in terms of distance between two disjoint closed sets. The precise results are as follows.

Theorem A: *Suppose (X, d) is a pseudo metric space. Then (X, d) is uniformly continuous space if and only if $\bar{A} \cap \bar{B} = \emptyset \Rightarrow d(A, B) > 0$.*

Theorem B: *Pseudo metric space (X, d) is uniformly continuous if and only if $\bar{A} \cap \bar{B} = \emptyset \Rightarrow \exists\, \alpha > 0$ such that $S_\alpha(A) \cap S_\alpha(B) = \emptyset$.*

We also obtain the sufficient condition for a pseudo metric space to be uniformly continuous. It is proved that

Theorem C: *If (X, d) is a pseudo metric space such that for any two disjoint closed sets at least one is compact, then (X, d) is uniformly continuous space.*

Keywords: *Uniformly continuous space, Uniformly continuous function, Pseudo Metric Space.*

INTRODUCTION

Let X be a pseudo metric space with pseudo metric d. Let A be any non empty subset of a pseudo metric space X. Then $S_\alpha(A) = \{ x \in X : d(x, y) < \alpha$ for some $y \in A\}$, where α is any positive real number. Also for any A, B subsets of X, we shall denote by $d(A, B)$, the distance between two sets A and B i.e. $d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$.

Firstly we show that in a pseudo metric space the distance between two sets is equal to the distance between their closures.

Lemma1: Let (X, d) be a pseudo metric space. Suppose $A, B \subset X$ then $d(A, B) = d(\bar{A}, \bar{B})$.

Proof: As $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$ then

$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$$
$$\geq \inf\{d(a, b) : a \in \bar{A} \text{ and } b \in \bar{B}\}$$
$$= d(\bar{A}, \bar{B})$$

i.e. $d(A, B) \geq d(\bar{A}, \bar{B})$. ....(1)

Now we show that $d(A, B) \leq d(\bar{A}, \bar{B})$.

Let $\varepsilon > 0$ be given.

Then $\exists\, a \in \bar{A}$ & $b \in \bar{B}$ such that $d(\bar{A}, \bar{B}) \leq d(a, b) < d(\bar{A}, \bar{B}) + \frac{\varepsilon}{3}$ ....(2)

Since $a \in \bar{A}$ & $b \in \bar{B}$, $\exists\, a' \in A$ & $b' \in B$ such that $d(a, a') < \frac{\varepsilon}{3}$ and $d(b, b') < \frac{\varepsilon}{3}$.

$$\therefore\ d(A, B) \leq d(a', b')$$
$$\leq d(a', a) + d(a, b) + d(b, b')$$
$$< \frac{\varepsilon}{3} + d(\bar{A}, \bar{B}) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad \text{from (2)}$$
$$= d(\bar{A}, \bar{B}) + \varepsilon$$

i.e. $d(A, B) < d(\bar{A}, \bar{B}) + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, $d(A, B) \leq d(\bar{A}, \bar{B})$ ....(3)

From (1) and (3) we get $d(A, B) = d(\bar{A}, \bar{B})$.

Theorem 2: Let (X, d) be a pseudo metric space. Suppose $A, B \subset X$. Then $d(\bar{A}, \bar{B}) > 0$ if and only if $\exists\, \alpha > 0$ such that $S_\alpha(A) \cap S_\alpha(B) = \emptyset$.

Proof: Suppose $d(\bar{A}, \bar{B}) > 0$. i.e. $\inf\{d(x, y) : x \in \bar{A} \text{ and } y \in \bar{B}\} > 0$.

Put $d(\bar{A}, \bar{B}) = r > 0$. Take $\alpha = \frac{r}{2}$.

We show that $S_\alpha(A) \cap S_\alpha(B) = \emptyset$.

Suppose, $S_\alpha(A) \cap S_\alpha(B) \neq \emptyset$. Then $\exists\, z \in S_\alpha(A) \cap S_\alpha(B)$

i.e. $z \in S_\alpha(A)$ and $z \in S_\alpha(B)$.

Thus there are $a \in A$ and $b \in B$ such that $d(z, a) < \alpha$ and $d(z, b) < \alpha$.

$\therefore\ d(a, b) \leq d(a, z) + d(z, b) < 2\alpha = r$.

i.e. $d(a, b) < r = d(\bar{A}, \bar{B})$ where $a \in \bar{A}$ and $b \in \bar{B}$.

This gives contradiction. Thus $S_\alpha(A) \cap S_\alpha(B) = \emptyset$.

Converse: Suppose $\exists\, \alpha > 0$ such that $S_\alpha(A) \cap S_\alpha(B) = \emptyset$. We show that $d(\bar{A}, \bar{B}) > 0$.

Suppose this is not true. i.e. $d(\bar{A}, \bar{B}) = 0$.

i.e. $inf\{d(a,b) : a \in \bar{A}, b \in \bar{B}\} = 0$.
Thus for each $\varepsilon > 0$, $\exists\, a \in \bar{A}$ & $b \in \bar{B}$ such that $d(a,b) < \varepsilon$
$\therefore$ for each $n \in \mathbb{N}, \exists\, a_n \in \bar{A}$ and $b_n \in \bar{B}$ such that $d(a_n, b_n) < \frac{1}{n}$.
Now since $a_n \in \bar{A}$ there is c_n in A such that $d(a_n, c_n) < \frac{1}{n}$ and since $b_n \in \bar{B}$ there is d_n in B such that $d(b_n, d_n) < \frac{1}{n}$.

$$\therefore d(c_n, d_n) \leq d(c_n, a_n) + d(a_n, b_n) + d(b_n, d_n) < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{3}{n} \to 0 \quad \text{as } n \to \infty$$

Since by assumption there is $\alpha > 0$ such that $S_\alpha(A) \cap S_\alpha(B) = \emptyset$, choose a positive integer N such that $\frac{3}{N} < \alpha$. For this N, $d(c_N, d_N) < \frac{3}{N} < \alpha$.
Thus $d_N \in B, C_N \in A, d_N \in S_\alpha(B)$ and $d_N \in S_\alpha(A)$.
Thus $d_N \in S_\alpha(A) \cap S_\alpha(B)$. i.e. $S_\alpha(A) \cap S_\alpha(B) \neq \emptyset$.
This gives contradiction.
$\therefore$ Our assumption that $d(\bar{A}, \bar{B}) = 0$ is wrong and hence □(□, □) > 0.
For the proof of the following theorem we require the Efremovic lemma as follows.

Lemma 3:
Efremovic lemma:
Let (x_n) and (y_n) be sequences in a metric space (X,d) such that for each n, $d(x_n, y_n) > \varepsilon$. Then there is an infinite set E of positive integers such that
$D_d(\{x_n, n \in E\}, \{\{y_n, n \in E\}) \geq \frac{\varepsilon}{4}$.
This is the lemma 3.3.1, p. 92[1].

Theorem 4:
Suppose (X,d) is a pseudo metric space. Then (X,d) is uniformly continuous space if and only if $\bar{A} \cap \bar{B} = \emptyset \Rightarrow d(A,B) > 0$.

Proof: Suppose pseudo metric space (X,d) is uniformly continuous.
We show that when $A, B \subset X$ satisfy $\bar{A} \cap \bar{B} = \emptyset$ then $d(A,B) > 0$.
By using lemma1 and theorem 2, it is enough to show that when $A, B \subset X$ satisfy $\bar{A} \cap \bar{B} = \emptyset$ then $\exists\, \alpha > 0$ such that $S_\propto(A) \cap S_\propto(B) = \emptyset$.
Let $A, B \subset X$ satisfy $\bar{A} \cap \bar{B} = \emptyset$.
Suppose $\forall\, \alpha > 0, S_\propto(A) \cap S_\propto(B) \neq \emptyset$.
Thus for each $n \in \mathbb{N}$, $S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B) \neq \emptyset$ and hence for each n, $\exists\, z_n \in S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B)$
$\therefore$ we get a sequence $\{z_n\}$ such that $z_n \in S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B) \quad \forall$ n.

Now for each n, $z_n \in S_{\frac{1}{n}}(A) \Rightarrow \exists\, x_n \in A$ such that $d(z_n, x_n) < \frac{1}{n}$.
Similarly, $z_n \in S_{\frac{1}{n}}(B) \Rightarrow \exists\, y_n \in B$ such that $d(z_n, y_n) < \frac{1}{n}$

$$\therefore d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

Now we define a function $f: X \to \mathbb{R}$ as $f(x) = \frac{d(x,\bar{A})}{d(x,\bar{A}) + d(x,\bar{B})}$, $x \in X$ …………(1)
Here for each n, $x_n \in A \Rightarrow x_n \in \bar{A}$
$\therefore d(x_n, \bar{A}) = 0 \Rightarrow f(x_n) = 0 \;\; \forall n$.
Also $y_n \in B \Rightarrow y_n \in \bar{B}$.
$\therefore d(y_n, \bar{B}) = 0$

$$\Rightarrow f(y_n) = \frac{d(y_n, \bar{A})}{d(y_n, \bar{A}) + d(y_n, \bar{B})} = \frac{d(y_n, \bar{A})}{d(y_n, \bar{A})} = 1$$

i.e. $\forall n, f(y_n) = 1$.
The function f defined in (1) is continuous on X.
Since (X,d) is uniformly continuous, the function f is also uniformly continuous.
i.e. for each $\varepsilon > 0, \exists\, \delta > 0$ such that $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.
Now for $\varepsilon = \frac{1}{2}$ $\exists\, \delta > 0$ such that $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
Choose $n_0 \in \mathbb{N}$ for which $\frac{2}{n_0} < \delta$
then $d(x_{n_0}, y_{n_0}) < \frac{2}{n_0} < \delta$
but $|f(x_n) - f(y_n)| = |0 - 1| = 1 > \varepsilon$
$\Rightarrow f$ is not uniformly continuous.
i.e. a continuous function $f: X \to \mathbb{R}$ is not uniformly continuous.
This gives contradiction. Thus our assumption is wrong and hence
when $A, B \subset X$ satisfy $\bar{A} \cap \bar{B} = \emptyset$ then $\exists\, \alpha > 0$ such that $S_\propto(A) \cap S_\propto(B) = \emptyset$.

Converse: Suppose $A, B \subset X$ with $\bar{A} \cap \bar{B} = \emptyset \Rightarrow d(A,B) > 0$.
We show that pseudo metric space (X,d) is uniformly continuous.
Suppose the mapping $f: X \to \mathbb{R}$ is continuous but not uniformly continuous.
i.e. $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$, $\exists\, x_\delta, y_\delta \in X$,
$d(x_\delta, y_\delta) < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$ ……….(2)
Thus $\forall n \in \mathbb{N}, \exists\, x_n, y_n \in X$, $d(x_n, y_n) < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ …………(3)
i.e. there exists sequences $\{x_n\}$ and $\{y_n\}$ in X with $d(x_n, y_n) < \frac{1}{n}$
but $|f(x_n) - f(y_n)| \geq \varepsilon \quad \forall\, n$.
By lemma3, for the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ in $\mathbb{R}$ there exists subsequences

$\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ respectively such that
$|f(x_{n_k}) - f(y_{n_l})| \geq \frac{\varepsilon}{4} \quad \forall\, k, l \geq 1.$
..........(4)

Denote $A = \{x_{n_k} : k \geq 1\}$ and $B = \{y_{n_l} : l \geq 1\}$.
Then $A \cap B = \emptyset$.
Now we show that $\bar{A} \cap \bar{B} = \emptyset$.
Suppose $x \in \bar{A} \cap \bar{B}$ then $x \in \bar{A}$ and $x \in \bar{B}$.
As f is continuous at a point x there exists $\delta' > 0$ such that
$d(x, y) < \delta' \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{8}$.
As $x \in \bar{A}$ and $x \in \bar{B}$ thus for δ' neighborhood of x there exist $x_{n_k} \in A$ and $y_{n_l} \in B$ such that $d(x_{n_k}, x) < \delta'$ and $d(y_{n_l}, x) < \delta'$.
$\Rightarrow |f(x_{n_k}) - f(x)| < \frac{\varepsilon}{8}$ and $|f(y_{n_l}) - f(x)| < \frac{\varepsilon}{8}$.
$\Rightarrow |f(x_{n_k}) - f(y_{n_l})| < \frac{\varepsilon}{4}$.
This gives contradiction to (4) and hence $\bar{A} \cap \bar{B} = \emptyset$.
Then by hypothesis, $d(A, B) > 0$. ....(5)

But $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$
$= \inf\{d(x_{n_k}, y_{n_l}) : x_{n_k} \in A, y_{n_l} \in B\}$
$\leq d(x_{n_k}, y_{n_k})$
$< \frac{1}{n_k} \to 0 \quad$ as $k \to \infty$

i.e. $d(A,B) = 0$ gives contradiction to (5).
Hence our assumption that "f is not uniformly continuous" is wrong.
Thus f is uniformly continuous.
Combining above lemma and theorem we get the characterization of uniformly continuous pseudo metric spaces.

Theorem 5:
Pseudo metric space (X, d) is uniformly continuous if and only if
$\bar{A} \cap \bar{B} = \emptyset \Rightarrow \exists\, \alpha > 0$ such that $S_\propto(A) \cap S_\propto(B) = \emptyset$.

Theorem 6:
Let X be a topological space. Consider the following conditions on X.
a] Every infinite subset of X has an ω-accumulation point.
b] Every sequence in X has a cluster point.
c] For each sequence in X there is a subsequence converging to a point of X.
d] The space X is compact.
These conditions are related as follows.
For all spaces [a] is equivalent to [b] and [d] implies [a].
If X satisfies the first axiom of countability, then [a], [b] and [c] are equivalent.
If X satisfies the second axiom of countability, then all four conditions are equivalent.
If X is pseudo metric space, then each of the four conditions implies that X satisfies the second axiom of countability and all four are equivalent.
This is the theorem 5, p. 138[2].
By using the above theorem, the following theorem gives the sufficient condition for (X, d) to be uniformly continuous space.

Theorem 7
If (X, d) is a pseudo metric space such that for any two disjoint closed sets at least one is compact, then (X, d) is uniformly continuous space.

Proof: We show that (X, d) is uniformly continuous space. By above theorem 5, it is enough to show that $\bar{A} \cap \bar{B} = \emptyset \Rightarrow \exists\, \alpha > 0$ such that $S_\propto(A) \cap S_\propto(B) = \emptyset$.
Suppose this is not true. i.e. $\bar{A} \cap \bar{B} = \emptyset$ but for any $\alpha > 0$, $S_\propto(A) \cap S_\propto(B) \neq \emptyset$.
$\bar{A} \cap \bar{B} = \emptyset$. Then by hypothesis at least one of them is compact. Suppose $\bar{A}$ is compact.
For any $\alpha > 0$, $S_\propto(A) \cap S_\propto(B) \neq \emptyset$. Thus for each $n \in \mathbb{N}$, $S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B) \neq \emptyset$.
So we may choose a point $z_n \in S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B) \quad \forall\, n \in \mathbb{N}$.
Thus we get a sequence $\{z_n\}$ such that $z_n \in S_{\frac{1}{n}}(A) \cap S_{\frac{1}{n}}(B) \quad \forall\, n$.
Now for each n, $z_n \in S_{\frac{1}{n}}(A)$ and $z_n \in S_{\frac{1}{n}}(B)$
$\Rightarrow \forall\, n,\ \exists\, a_n \in A\ \&\ b_n \in B$ such that $d(z_n, a_n) < \frac{1}{n}$ and $d(z_n, b_n) < \frac{1}{n}$
i.e. $d(z_n, a_n) \to 0$ and $d(z_n, b_n) \to 0$ as $n \to \infty$
Now $\bar{A}$ is compact thus it is sequentially compact by theorem 6. Hence there exists a subsequence $\{a_{n_k}\}$ in A converges to some $a \in \bar{A}$. i.e. $a_{n_k} \to a$
$\Rightarrow d(a_{n_k}, a) \to 0$ as $k \to \infty$.
Now we show that $b_{n_k} \to a$.
Let $\varepsilon > 0$ be given.
As $a_{n_k} \to a$ there exist $N_1 > 0$ such that $\forall\, k \geq N_1$, $d(a_{n_k}, a) < \frac{\varepsilon}{3}$.
Also, $d(a_n, z_n) \to 0 \quad$ as $n \to \infty$
$\Rightarrow \exists\, N_2 > 0$ such that $\forall\, k \geq N_2$, $d(a_{n_k}, z_{n_k}) < \frac{\varepsilon}{3}$.
Take $N = \max(N_1, N_2)$ then $\forall\, k \geq N$,
$d(z_{n_k}, a) \leq d(z_{n_k}, a_{n_k}) + d(a_{n_k}, a)$
$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$

Also $d(b_n, z_n) \to 0$ as $n \to \infty$
$\Rightarrow$ for the above $\varepsilon > 0$, $\exists\ N_3 > 0$ such that $\forall\ k \geq N_3,\ d(b_{n_k}, z_{n_k}) < \frac{\varepsilon}{3}$.
Take $M = max\,(N, N_3)$ then $\forall\ k \geq M$,

$$d\left(b_{n_k}, a\right) \leq d\left(b_{n_k}, z_{n_k}\right) + d\left(z_{n_k}, a\right) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

i.e. $b_{n_k} \to a$ as $k \to \infty$.
But $\{b_{n_k}\}$ is a sequence in $\bar{B}$ and $\bar{B}$ is closed. Thus $a \in \bar{B}$ and therefore $a \in \bar{A} \cap \bar{B}$.
Hence $\bar{A} \cap \bar{B} \neq \emptyset$. This gives a contradiction. Thus our assumption is wrong.
Hence $\exists\ \alpha > 0$ such that $S_{\propto}(A) \cap S_{\propto}(B) = \emptyset$.

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