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Box Dimensions of Two Dimensional Cantor like Sets and Special Property of Cantor like Sets Similar to Sets of Positive Measure

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ABSTRACT: In this paper we determine the estimates for upper / lower Box dimension of two dimensional generalized Cantor like sets S. The two dimensional Cantor like sets S have been constructed in [3], for any sequence $\{\epsilon_n\}$ with $0 < \epsilon_n < 1$ with the help of sequence of sets $\{E_n\}$ of subsets of [0, 1]× [0, 1] such that $E_n \supset E_{n+1}$, S = $\cap E_n$ and m(S) = $[\prod_{n=1}^{\infty}(1-\epsilon_n)]^2$. Further if $\sum \epsilon_n < \infty$, m(S) ≠ 0 and if $\sum \epsilon_n = \infty$, m(S) = 0.

In particular it is proved that the upper and lower Box dimension of the set S satisfy $\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\,[\epsilon_{k+1}]^{1/k}\}}$ and $\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$. Further we generalize the property of sets of positive measure to Cantor like sets namely " If T is a measurable set of positive measure then T^* = { x – y : x ∈ T , y ∈ T } = T – T contains an interval ($-\alpha$, α) for some $\alpha > 0$". We in fact prove that if S is Cantor like subset of [0, 1] then S^* = S – S = [-1, 1].

KEYWORDS: Cantor set, Cantor like set, upper Box dimension and lower Box dimension.

I. INTRODUCTION

Definition 1.1 Box Counting Dimension

Let F be any nonempty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower and upper box dimensions of F are defined respectively as $\underline{dim}_B(F) = \underline{lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$ and $\overline{dim}_B(F) = \overline{lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$.

If the lower and upper box-counting dimensions of F are equal then the common value is called box-counting dimension of F or box- dimension of F and $dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$.

Above definition is taken from Kenneth Falconer [2].

II. MAIN RESULTS

A. CONSTRUCTION OF TWO DIMENSIONAL GENERALIZED CANTOR LIKE SET

At first stage we divide unit square E_0 into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{1-\epsilon_1}{2}$, 4 are rectangles each of size $\epsilon_1 \times(\frac{1-\epsilon_1}{2})$ and 1 centre square is of side ϵ_1.

Let E_1 be a set obtained by removing middle rectangle from each column and each row. Thus E_1 is the union of 4 corner squares each of side $\frac{1-\epsilon_1}{2}$.

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At second stage we divide each remaining square again into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$, 4 are rectangles each of length $\frac{(1-\epsilon_1)\,\epsilon_2}{2}$ and breadth $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$ and 1 centre square is of side $\frac{(1-\epsilon_1)\,\epsilon_2}{2}$.

Let E_2 be a set obtained by removing middle rectangle from each column and each row from each remaining square. Thus E_2 is union of 4^2 squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)}{4}$.

At third stage we divide each remaining squares again into three columns and three rows consisting of nine rectangles out of which 4 are corner squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$, 4 are rectangles each of length $\frac{(1-\epsilon_1)(1-\epsilon_2)\,\epsilon_3}{4}$ and breadth $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$ and 1 centre square is of side $\frac{(1-\epsilon_1)(1-\epsilon_2)\,\epsilon_3}{4}$.

Let E_3 be a set obtained by removing middle rectangle from each column and each row from each remaining square. Thus E_3 is union of 4^3 squares each of side $\frac{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}{8}$.

Continuing in this way at k^{th} stage E_k consists of union of 4^k squares each of side $\prod_{i=1}^{k}\frac{(1-\epsilon_i)}{2}$.

We define

$$S = \bigcap_{k=0}^{\infty} E_k$$

Where $E_k = \cup J_{j,k}$, j = 1, 2, …. , 4^k, each $J_{j,k}$ being a square of side $\prod_{i=1}^{k}\frac{(1-\epsilon_i)}{2}$.

Above construction is from [3].

We determine of Box dimension of two dimensional generalized Cantor like set constructed in [3].

Theorem 2.2

Lower and upper box-dimensions of two dimensional generalized Cantor like set S satisfy $\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\,[\epsilon_{k+1}]^{1/k}\}}$ and $\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty}\frac{\log 4}{-\log\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$.

Proof

We know that in the construction of two dimensional generalized Cantor like set S, at k^{th} stage the set E_k consists of union of 4^k disjoint squares each of diameter $\leq \sqrt{2}\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]$. Take $\delta_k = \sqrt{2}\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]$. Then $\delta_k \to 0$ as $k \to \infty$.

$\therefore$ At most 4^k squares of diameter $\leq \delta_k$ are required to cover S.

$$\therefore N_{\delta_k}(S) \leq 4^k$$

By definition 1.1 we get,

$$\overline{dim}_B(S) = \overline{lim}_{\delta_k\to 0}\frac{\log N_{\delta_k}(S)}{-\log\delta_k}$$ .Since $\delta_k \to 0$ as $k \to \infty$., we have

$$\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty}\frac{\log 4^k}{-\log\{\sqrt{2}\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]\}}$$

$$\leq \overline{lim}_{k\to\infty}\frac{\log 4^k}{-\log\{\sqrt{2}\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\}^{k/k}}$$

$$= \overline{lim}_{k\to\infty}\frac{k\log 4}{-k\log\{\sqrt{2}\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\}^{1/k}}$$

$$= \overline{lim}_{k\to\infty}\frac{\log 4}{-\log\{\sqrt{2}\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\}^{1/k}} = \overline{lim}_{k\to\infty}\frac{\log 4}{-\log[\sqrt{2}]^{1/k}-\log[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$$

$$\therefore \quad \overline{dim}_B(S) \leq \overline{lim}_{k\to\infty}\frac{\log 4}{-\log[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$$

Now we show that $\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\,[\epsilon_{k+1}]^{1/k}\}}$, consider a rectangle U of diameter $\leq \delta, \delta > 0$ and choose k such that

$\sqrt{2}\,\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\} \leq \delta < \sqrt{2}\{[\prod_{n=1}^{k-1}\frac{(1-\epsilon_n)}{2}]\epsilon_k\}$, then U intersects at most one of the basic square of diameter $\leq \sqrt{2}\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]$ used in the construction of S.

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For if U intersects more than one square of E_k then $|U| \geq$ distance between two disjoint squares of E_k i.e. $|U| \geq \sqrt{2}\{[\prod_{n=1}^{k-1}\frac{(1-\epsilon_n)}{2}]\epsilon_k\}$,

i.e. $\delta > \sqrt{2}\{[\prod_{n=1}^{k-1}\frac{(1-\epsilon_n)}{2}]\epsilon_k\}$,

which is false because $\delta < \sqrt{2}\{[\prod_{n=1}^{k-1}\frac{(1-\epsilon_n)}{2}]\epsilon_k\}$.

These are 4^k such squares.

$\therefore$ Atleast 4^k squares of diameter $\leq \delta$ are required to cover $E_k \supset S$.

$$\therefore N_\delta(S) \geq 4^k$$

Now $\delta \geq \sqrt{2}\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\}$

$$\therefore \frac{1}{-\log\delta} = \frac{1}{\log(\frac{1}{\delta})} \geq \frac{1}{-\log\{\sqrt{2}\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\}\}}$$

By definition 1.1 we get,

$$\underline{dim}_B(S) = \underline{lim}_{\delta\to 0}\frac{\log N_\delta(S)}{-\log\delta}$$

$$\geq \underline{lim}_{k\to\infty}\frac{\log 4^k}{-\log\{\sqrt{2}\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\}\}}$$

$$= \underline{lim}_{k\to\infty}\frac{\log 4^k}{-\log\{\sqrt{2}\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\}\}^{k/k}}$$

$$= \underline{lim}_{k\to\infty}\frac{k\log 4}{-k\log\{\sqrt{2}\{\left[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}\right]\epsilon_{k+1}\}\}^{1/k}}$$

$$= \underline{lim}_{k\to\infty}\frac{\log 4}{-\log[\sqrt{2}]^{1/k}\ -\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]\epsilon_{k+1}\}^{1/k}}$$

$$\therefore \quad \underline{dim}_B(S) \geq \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}[\epsilon_{k+1}]^{1/k}\}}$$

Now we take particular cases corresponding to different values of $\epsilon_n = \epsilon$, a constant , $0 < \epsilon < 1$, in Theorem 2.2 which gives Box dimension of corresponding two dimensional Cantor like set.

Particular Case 2.3

When $\epsilon_n = \frac{1}{5}$, in Theorem 2.2 box dimension of two dimensional Cantor - $\frac{2}{5}$ set is $\frac{\log 4}{\log(\frac{5}{2})}$.

Let $\epsilon_n = \frac{1}{5}$

$$\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\ [\epsilon_{k+1}]^{1/k}\}}$$

$$= \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\frac{1}{5})}{2}]^{1/k}\ [\frac{1}{5}]^{1/k}\}} = \underline{lim}_{k\to\infty}\frac{\log 4}{-\log\{(\frac{2}{5})\ [\frac{1}{5}]^{1/k}\}} = \frac{\log 4}{\log(\frac{5}{2})}.$$

$$\therefore \quad \underline{dim}_B(S) \geq \frac{\log 4}{\log(\frac{5}{2})} \qquad \text{.....(1)}$$

Now, $\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty}\frac{\log 4}{-\log\ [\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$

$$= \overline{lim}_{k\to\infty}\frac{\log 4}{-\log\ [\prod_{n=1}^{k}\frac{(1-\frac{1}{5})}{2}]^{1/k}} = \overline{lim}_{k\to\infty}\frac{\log 4}{-\log\ [[\frac{2}{5}]^k]^{1/k}} = \frac{\log 4}{\log(\frac{5}{2})}.$$

$$\therefore \overline{dim}_B(S) \leq \frac{\log 4}{\log(\frac{5}{2})} \qquad \text{.....(2)}$$

From equations (1) and (2) we get,

$$dim_B(S) = \frac{\log 4}{\log(\frac{5}{2})}.$$

which is box dimension of two dimensional Cantor - $\frac{2}{5}$ set.

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Particular Case 2.4

When $\epsilon_n = \frac{1}{7}$, in Theorem 2.2 box dimension of two dimensional Cantor - $\frac{3}{7}$ set is $\frac{\log 4}{\log(\frac{7}{3})}$.

Let $\epsilon_n = \frac{1}{7}$,

$$\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\ [\epsilon_{k+1}]^{1/k}\}}$$

$$= \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\frac{1}{7})}{2}]^{1/k}\ [\frac{1}{7}]^{1/k}\}} = \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{(\frac{3}{7})\ [\frac{1}{7}]^{1/k}\}} = \frac{\log 4}{\log(\frac{7}{3})}.$$

$$\therefore \quad \underline{dim}_B(S) \geq \frac{\log 4}{\log(\frac{7}{3})} \qquad \text{.....(3)}$$

Now

$$\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}} = \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [\prod_{n=1}^{k}\frac{(1-\frac{1}{7})}{2}]^{1/k}}$$

$$= \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ \{[2]^{1/k}\ [(\frac{3}{7})^k]^{1/k}\}} = \frac{\log 4}{\log(\frac{7}{3})}.$$

$$\therefore\ \overline{dim}_B(S) \leq \frac{\log 4}{\log(\frac{7}{3})} \qquad \text{.....(4)}$$

From equations (3) and (4) we get,

$$dim_B(S) = \frac{\log 4}{\log(\frac{7}{3})}.$$

which is box dimension of two dimensional Cantor - $\frac{3}{7}$ set.

Particular Case 2.5

When $\epsilon_n = \frac{1}{2}$, in Theorem 2.2 box dimension of two dimensional Cantor - $\frac{1}{4}$ set is 1.

Let $\epsilon_n = \frac{1}{2}$,

$$\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\ [\epsilon_{k+1}]^{1/k}\}}$$

$$= \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\frac{1}{2})}{2}]^{1/k}\ [\frac{1}{2}]^{1/k}\}} = \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{(\frac{1}{4})\ [\frac{1}{2}]^{1/k}\}} = \frac{\log 4}{\log 4} = 1$$

$$\therefore \quad \underline{dim}_B(S) \geq 1 \qquad \text{.....(5)}$$

Now

$$\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [2\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$$

$$= \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [2\prod_{n=1}^{k}\frac{(1-\frac{1}{2})}{2}]^{1/k}} = \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ \{[2]^{1/k}\ [(\frac{1}{4})^k]^{1/k}\}} = 1.$$

$$\therefore\ \overline{dim}_B(S) \leq 1 \qquad \text{.....(6)}$$

From equations (5) and (6) we get,

$$dim_B(S) = 1$$

which is box dimension of two dimensional Cantor - $\frac{1}{4}$ set.

Remark 2.6

If $\epsilon_n = \epsilon$, a constant , $0 < \epsilon < 1$, then the estimates for lower and upper box-dimensions of two dimensional generalized Cantor like set S determined in Theorem 2.2 are equal and the common value is $\frac{\log 4}{-\log(\frac{1-\epsilon}{2})}$.

Proof

Let $\epsilon_n = \epsilon$,

$$\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\ [\epsilon_{k+1}]^{1/k}\}}$$

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$$= \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon)}{2}]^{1/k}\ [\epsilon]^{1/k}\}} = \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{(\frac{(1-\epsilon)}{2})\ [\epsilon]^{1/k}\}} = \frac{\log 4}{-\log(\frac{1-\epsilon}{2})}$$

$$\therefore \quad \underline{dim}_B(S) \geq \frac{\log 4}{-\log(\frac{1-\epsilon}{2})} \qquad \text{.....(1)}$$

Now

$$\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [2\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$$

$$= \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ [2\prod_{n=1}^{k}\frac{(1-\epsilon)}{2}]^{1/k}} = \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\ \{[2]^{1/k}\,[(\frac{1-\epsilon}{2})^{k}]^{1/k}\}} = \frac{\log 4}{-\log(\frac{1-\epsilon}{2})}.$$

$$\therefore \ \overline{dim}_B(S) \leq \frac{\log 4}{-\log(\frac{1-\epsilon}{2})} \qquad \text{.....(2)}$$

From equations (1) and (2) we get,

$$dim_B(S) = \frac{\log 4}{-\log(\frac{1-\epsilon}{2})}$$

Now we generalize the following property of sets of positive measure, to Cantor like sets. " If T is a measurable set of positive measure then T^* = { x – y : x ∈ T , y ∈ T } = T – T contains an interval ($-\alpha$, α) for some α > 0".

This result is from G. de. Barra [1], p – 43, Example 20.

In fact we prove that if S is a Cantor like set even if their measures are zero they satisfy the property that S^* = S – S contains an interval. The precise statement of the theorem is as follows.

Theorem

If S is one of the Cantor like sets then S^* = S – S = [-1 ,1].

This is proved in the form of Lemma and Theorem as follows.

Lemma 2.7

If x ∈ [0 ,1] and x has finite expansion in base 5 of length n then there exists b , c of Cantor - $\frac{1}{5}$ set such that x = b – c and expansion of b in base 5 is not longer than that of a and expansion of c in base 5 is either no longer than that of a or all the digits from n + 1 onwards are 4.

Proof

We prove this result by induction

Let P(n) : If x ∈ [0 ,1] and x has finite expansion in base 5 of length n then there b , c of Cantor - $\frac{1}{5}$ set such that x = b – c and expansion of b in base 5 is not longer than that of a and expansion of c in base 5 is either no longer than that of a or all the digits from n + 1 onwards are 4.

For n = 1, there are just five reals of length 1.

∴ x = 0·0 = 0·0 - 0·0

$$x = 0\cdot1 = \frac{1}{5} = \frac{2}{5} - \frac{1}{5} = \frac{2}{5} - \frac{\frac{4}{3^2}}{1-\frac{1}{5}} = \frac{2}{5} - (\frac{0}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots..) = 0\cdot2 - 0\cdot0444\ldots.$$

x = 0·2 = 0·2 - 0·0

$$x = 0\cdot3 = \frac{3}{5} = \frac{4}{5} - \frac{1}{5} = \frac{4}{5} - \frac{\frac{4}{3^2}}{1-\frac{1}{5}} = \frac{4}{5} - (\frac{0}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \ldots..) = 0\cdot4 - 0\cdot0444\ldots.$$

x = 0·4 = 0·4 - 0·0

∴ The result is true for n = 1.

Let the result be true for n-1.

We prove that it is true for n

Let $x = 0\cdot a_1a_2 \ldots..a_n = 0\cdot a_1a_2 \ldots..a_{n-1}\cdot a_n$, where $a_i \in \{0, 1, 2, 3, 4\}$, i = 1, 2, 3, …..

Since the result is true for n-1

∴ We can write

(i) $0\cdot a_1a_2 \ldots..a_{n-1} = 0\cdot b_1b_2 \ldots..b_{n-1} - 0\cdot c_1c_2 \ldots..c_{n-1}$ or

(ii) $0\cdot a_1a_2 \ldots..a_{n-1} = 0\cdot b_1'b_2' \ldots..b_{n-1}' - 0\cdot c_1'c_2' \ldots..c_{n-1}'444\ldots..$

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There are five cases where a_n = 0, 1, 2, 3, 4.

Case I $a_n = 0$

For (i) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}0$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1}0 - 0\cdot c_1c_2 \ldots\ldots c_{n-1}0$

$= b - c$

Where b, c satisfy required conditions.

For (ii) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}0$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots$

$= b' - c'$

Where b' , c' satisfy required conditions.

Case II For $a_n = 1$

For (i) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}1$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 1^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 1^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 2^{n^{th}place} - 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.$

$(\because 0\cdot0 \ldots\ldots 1^{n^{th}place} = \frac{1}{5^n} = \frac{2}{5^n} - \frac{1}{5^n} = \frac{2}{5^n} - \frac{4/5^{n+1}}{1-1/5} = \frac{2}{5^n} - \left(\frac{0}{5^n} + \frac{4}{5^{n+1}} + \frac{4}{5^{n+2}} + \ldots\ldots\right)$

$= 0\cdot0 \ldots\ldots 2^{n^{th}place} - 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.)$

$x = (0\cdot b_1b_2 \ldots\ldots b_{n-1} + 0\cdot0 \ldots\ldots 2^{n^{th}place}) - (0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.)$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1}\,2 - 0\cdot c_1c_2 \ldots\ldots c_{n-1}0444\ldots..$

$= b - c$

Where b, c satisfy required conditions.

For (ii) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}1$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 1^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots + 0\cdot0 \ldots\ldots 00^{n^{th}place}444\ldots.$

$(\because 0\cdot0 \ldots\ldots 1^{n^{th}place} = \frac{1}{5^n} = \frac{1}{5^n} = \frac{4/5^{n+1}}{1-1/5} = \frac{0}{5^n} + \frac{4}{5^{n+1}} + \frac{4}{5^{n+2}} + \ldots\ldots = 0\cdot0 \ldots 4^{(n+1)^{th}place}444\ldots.)$

$x = 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - (0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots - 0\cdot0 \ldots\ldots 00^{n^{th}place}444\ldots.)$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'400\ldots\ldots$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'4$

$x = b' - c'$

Where b' , c' satisfy required conditions.

Case III For $a_n = 2$

For (i) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}2$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 2^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 02^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1}\,2 - 0\cdot c_1c_2 \ldots\ldots c_{n-1}$

$= b - c$

Where b, c satisfy required conditions.

For (ii) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}2$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 2^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots + 0\cdot0 \ldots\ldots 2^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}'2 - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots$

$x = b' - c'$

Where b' , c' satisfy required conditions.

Case IV For $a_n = 3$

For (i) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}3$

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International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 3^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 3^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 4^{n^{th}place} - 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.$

$(\because 0\cdot0 \ldots\ldots 3^{n^{th}place} = \frac{3}{5^n} = \frac{4}{5^n} - \frac{1}{5^n} = \frac{4}{5^n} - \frac{4/5^{n+1}}{1-1/5} = \frac{4}{5^n} - \left(\frac{0}{5^n} + \frac{4}{5^{n+1}} + \frac{4}{5^{n+2}} + \ldots\ldots\right)$

$= 0\cdot0 \ldots\ldots 4^{n^{th}place} - 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.)$

$x = (0\cdot b_1b_2 \ldots\ldots b_{n-1} + 0\cdot0 \ldots\ldots 4^{n^{th}place}) - (0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.)$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1}\,4 - 0\cdot c_1c_2 \ldots\ldots c_{n-1}0444\ldots..$

$= b - c$

Where b, c satisfy required conditions.

For (ii) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}3$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 3^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots + 0\cdot0 \ldots\ldots 2^{n^{th}place} + 0\cdot0 \ldots\ldots 00^{n^{th}place}444\ldots.$

$(\because 0\cdot0 \ldots\ldots 3^{n^{th}place} = \frac{3}{5^n} = \frac{2}{5^n} + \frac{1}{5^n} = \frac{2}{5^n} + \frac{4/5^{n+1}}{1-1/5} = \frac{2}{5^n} + \left(\frac{0}{5^n} + \frac{4}{5^{n+1}} + \frac{4}{5^{n+2}} + \ldots\ldots\right)$

$= 0\cdot0 \ldots\ldots 2^{n^{th}place} + 0\cdot0 \ldots\ldots 4^{(n+1)^{th}place}444\ldots.)$

$x = 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' + 0\cdot0 \ldots\ldots 2^{n^{th}place} - (0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots - 0\cdot0 \ldots 00^{n^{th}place}444\ldots.)$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}'2 - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'400\ldots..$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}'2 - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'4$

$x = b' - c'$

Where b', c' satisfy required conditions.

Case V For $a_n = 4$

For (i) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}4$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 4^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1} - 0\cdot c_1c_2 \ldots\ldots c_{n-1} + 0\cdot0 \ldots\ldots 04^{n^{th}place}$

$= 0\cdot b_1b_2 \ldots\ldots b_{n-1}4 - 0\cdot c_1c_2 \ldots\ldots c_{n-1}$

$= b - c$

Where b, c satisfy required conditions.

For (ii) $x = 0\cdot a_1a_2 \ldots\ldots a_{n-1}4$

$= 0\cdot a_1a_2 \ldots\ldots a_{n-1} + 0\cdot0 \ldots\ldots 4^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}' - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots + 0\cdot0 \ldots\ldots 4^{n^{th}place}$

$= 0\cdot b_1'b_2' \ldots\ldots b_{n-1}'4 - 0\cdot c_1'c_2' \ldots\ldots c_{n-1}'444\ldots\ldots$

$x = b' - c'$

Where b', c' satisfy required conditions.

∴ Result is true for P(1) and P(n-1) ⟹ P(n)

∴ Result is true for all n.

∴ For any finite fraction x having expansion in base 5, we can find two elements b and c from Cantor - $\frac{1}{5}$ set such that x = b – c.

Theorem 2.8

i) $[0,1] \subset S - S$

ii) $[-1,1] = S - S$, where S is Cantor - $\frac{1}{5}$ set i.e. S = { x ∈ [0,1] : x has base 5 expansion containing 0, 2, 4 only}

Proof i) We show that $[0,1] \subset S - S$, where S is Cantor - $\frac{1}{5}$ set. i.e. S = { x ∈ [0,1] : x has base 5 expansion containing 0, 2, 4 only}

Let a ∈ [0,1] and $a = 0\cdot a_1a_2\ldots\ldots$ be a base 5 expansion of a.

If $x_k = 0\cdot a_1a_2\ldots\ldots a_k$, $k \geq 1$. Then $a = \lim_{i\to\infty} x_i$.

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(An ISO 3297: 2007 Certified Organization)

Vol. 5, Issue 12, December 2016

From Lemma 2.7 for k ≥ 1 there exist members of the Cantor- $\frac{1}{5}$ set b_k and $c_k$such that $x_k = b_k - c_k$.

Since $\{b_i\} \subset S$, where S is compact there is a convergent subsequence $\{b_{ij}\}$ which converges. Similarly corresponding to $\{b_{ij}\}$ there is a sequence $\{c_{ij}\}$ which has a convergent subsequence $\{c_{ijr}\}$.

Since each subsequence must converge to the same limit as that of its parent sequence, subsequence $\{b_{ijr}\}$ and $\{c_{ijr}\}$ are also convergent. Let their limits be b and c respectively, where b, c ∈ S.

$$\text{Also } a = \lim_{i\to\infty} x_i$$
$$= \lim_{r\to\infty} x_{ijr}$$
$$= \lim_{r\to\infty} (b_{ijr} - c_{ijr})$$
$$= b - c$$
$$\therefore a = b - c, \text{ where } b, c \in S$$
$$\therefore a \in S - S$$
$$\therefore a \in [0, 1] \Longrightarrow a \in S - S$$
$$[0, 1] \subset S - S$$

ii) We show that [-1 ,1] = S- S, where S is Cantor - $\frac{1}{5}$ set.

By part (i) we know that [0 ,1] ⊂ S- S …… (1)

We show that [-1 ,0] ⊂ S- S

Let x ∈ [-1 ,0]

∴ - x ∈ [0, 1]

∴ - x = b – c , where b, c ∈ S (By part (i))

∴ x = c – b , where b, c ∈ S

∴ x ∈ S- S

∴ x ∈ [-1 ,0] ⟹ x ∈ S- S

[-1 ,0] ⊂ S- S …… (2)

From equations (1) and (2), we get

[-1 ,1] ⊂ S- S …… (3)

Now we show that S- S ⊂ [-1 ,1]

Let x, y ∈ S

∴ 0 ≤ x ≤ 1 and 0 ≤ y ≤ 1

∴ 0 ≤ x ≤ 1 and - 1 ≤ - y ≤ 0

∴ - 1 ≤ x - y ≤ 1

∴ x – y ∈ [-1 ,1]

∴ S- S ⊂ [-1 ,1] …….. (4)

From equations (3) and (4), we get

[- 1 ,1] = S- S, where S is Cantor - $\frac{1}{5}$ set

Similar proofs can be given for Cantor ternary set, Cantor - $\frac{1}{7}$ set, Cantor - $\frac{1}{2n+1}$ set, n≥ 1.

Lemma 2.9

If x ∈ [0, 1] and x has finite ternary expansion of length n then there exists b, c of Cantor ternary set such that x = b – c and expansion of b is not longer than that of a, expansion of c is either no longer than that of a or all the digits from n + 1 onwards
are 2.

Theorem 2.10

i) $[0, 1] \subset S - S$

ii) [- 1 ,1] = S- S, where S is Cantor ternary set.

Lemma 2.11

If x ∈ [0, 1] and x has finite expansion in base 7 of length n then there exists b, c of Cantor - $\frac{1}{7}$ set such that x = b – c and expansion of b in base 7 is not longer than that of a and expansion of c in base 7 is either no longer than that of a or all the digits from

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n + 1 onwards are 6.

Theorem 2.12

i) [0 ,1] $\subset$ S − S

ii) [- 1 ,1] = S- S, where S is Cantor - $\frac{1}{7}$ set.

III. CONCLUSION

We have obtained estimates for for upper / lower Box dimension of such two dimensional generalized Cantor like sets S. In particular it is proved that the upper and lower Box dimension of the set S satisfy $\underline{dim}_B(S) \geq \underline{lim}_{k\to\infty} \frac{\log 4}{-\log\{[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}\,[\epsilon_{k+1}]^{1/k}\}}$ and $\overline{dim}_B(S) \leq \overline{lim}_{k\to\infty} \frac{\log 4}{-\log\,[\prod_{n=1}^{k}\frac{(1-\epsilon_n)}{2}]^{1/k}}$ and for the particular case $\epsilon_n = \epsilon$, a constant , $0 < \epsilon < 1$ the equality holds namely $dim_B(S) = \frac{-\log 4}{\log(\frac{1-\epsilon}{2})}$,. Also we observe that the property $S^* = S - S$ contains an interval is also satisfied by Cantor - $\frac{1}{5}$ set, Cantor - $\frac{1}{7}$ set even though they have measure zero.

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